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NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Annual Meeting, Richmond and Williamsburg, Va., December 27–31, 1938.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1938 and reported to the Secretary.

Allegheny Mountain, May 14; Oct. 22.

Illinois, Carbondale, May 13–14.

Indiana, Terre Haute, May 6–7.

Iowa, Sioux City, April 15–16.

Kansas, Pittsburg, April 2.

Kentucky, Morehead, May 14.

Louisiana-Mississippi, Starkville, Miss., March 11–12.

Maryland-District of Columbia-Virginia, Annapolis, Md., May 7; College Park, Md., December 10.

Michigan, Ann Arbor, March 19.

Minnesota, Collegeville, May 14; Minneapolis, October 28.

Missouri, Rolla, April 23.

Nebraska, Hastings, May 6.

Ohio, Columbus, March 31.

Oklahoma, Oklahoma City, February 11.

Philadelphia, Collegeville, Pa., Nov. 26.

Rocky Mountain, Boulder, Colo., April 15–16.

Southeastern, Atlanta, Ga., April 1–2.

Southern California, Claremont, March 26.

Southwestern, Albuquerque, N.M., April 25–26

Texas, Fort Worth, April 22.

Wisconsin, West De Pere, May 14.

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The National Council of Teachers of Mathematics

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The usual summer meeting of the Mathematical Association of America was omitted in 1938 in deference to the Semicentennial Celebration of the American Mathematical Society. The Trustees of the Association met at New York on Tuesday evening, September 6, 1938.

The following fifty-eight persons were elected to membership on applications duly certified:

L. V. Ahlfors, Ph.D.(Helsingfors) Prof., Univ. of Helsingfors, Helsingfors, Finland.

Harriet W. Allen, Ph.D.(Mass. Inst. of Tech.) Asst. Prof., Hollins Coll., Hollins, Va.

A. A. Aucoin, A.M.(Rice) Asst., Louisiana State Univ., University, La.

Mrs. Kathryn C. Baker, M.S.(Chicago) Instr., Baylor Univ., Waco, Texas.

W. A. Barnett, A.B.(W. Kentucky State Teachers Coll.) Teacher, High School, Burlington, Ky.

Edna C. Battin, A.M.(Southern California) Teacher, Union High School, Colton, Calif.

Jacob Bearman, A.B.(Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn.

E. A. Beito, A.M.(Minnesota) Asst. Prof., Univ. of Wichita, Wichita, Kans.

Samuel Borofsky, Ph.D.(Columbia) Asst. Prof., Brooklyn Coll., Brooklyn, N. Y.

J. A. Cooley, Ph.D.(Illinois) Asso. Prof., Univ. of Tennessee, Knoxville, Tenn.

T. M. Edison, B.S.(Mass. Inst. of Tech.) President, and research engineer, Calibron Products, Inc., West Orange, N. J.

H. C. T. Eggers, Ph.D.(Michigan) Asst. Prof., Drawing and desc. geom., Univ. of Minnesota, Minneapolis, Minn.

L. Beatrice Fleming, A.M.(Michigan) Instr., Florida A. and M. Coll., Tallahassee, Fla.

M. G. Galbraith, M.S.(Rutgers) Asst. Prof., Rutgers Univ., New Brunswick, N. J.

Carmelita T. González, M.S.(Tulane) Instr., William Woods Coll., Fulton, Mo.

H. J. G. Gutekunst, M.S.(Illinois) Teacher, Batavia High School, Batavia, Ill.

D. W. Hall, Ph.D.(Virginia) National Research Fellow, Univ. of Pennsylvania, Philadelphia Pa.

H. H. Hartzler, Ph.D.(Rutgers) Asso. Prof., Goshen Coll., Goshen, Ind.

Corinne R. Hattan, A.M.(Kansas) Head of Dept., Northwestern Jr. Coll., Orange City, Iowa.

Rev. J. R. Hearn, A.M.(Woodstock) Teacher, Coll. of St. Ignatius, Manhasset, N. Y.

E. R. Heineman, A.M.(Wisconsin) Asst. Prof., Texas Tech. Coll., Lubbock, Tex.

W. N. Herr, Ph.D.(Chicago) Instr., Chemistry, Jr. Coll., Duluth, Minn.

G. B. Huff, Ph.D.(Illinois) Asst. Prof., Southern Methodist Univ., Dallas, Tex.

J. F. Kenney, A.M.(Michigan) Instr., Northwestern Univ., Evanston, Ill.

Cornelius Lanczos, Ph.D.(Szeged) Prof., Purdue Univ., Lafayette, Ind.

Nathan Lazar, Ph.D.(Columbia) Teacher, Alexander Hamilton High School, Brooklyn, N. Y.

W. G. Leavitt, A.M.(Nebraska) Grad. asst., Univ. of Nebraska, Lincoln, Nebr.

Herman Levy, A.B.(New York Univ.) 3736 Oceanic Ave., Brooklyn, N. Y.

Neil Little, A.M.(Michigan) Instr., Purdue Univ., Lafayette, Ind.

Olwen Lloyd (Mrs. George), M.A.(Cambridge, England) Headmistress, Mount Vernon Seminary, Washington, D. C.

Margaret P. Martin, B.S.(Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn.

W. R. McEwen, B.E.(State Teachers Coll., Duluth) Asst., Univ. of Minnesota, Minneapolis, Minn.

Parry Moon, M.S.(Mass. Inst. of Tech.) Asso. Prof., Elec. Eng., Massachusetts Inst. of Tech., Cambridge, Mass.

Lillian Moore, A.M.(Pennsylvania) Teacher, Far Rockaway High School, Far Rockaway, N. Y.

Frederick Mosteller. Senior, Carnegie Inst. of Tech., Pittsburgh, Pa.

P. M. Niersbach, A.M.(Southern California) Teacher, Bell High School, Huntington Park, Calif.

Irene A. Nolan, B.S.(Marquette) Grad. student, Louisiana State Univ., University, La.

Llewellyn Notley, A.M.(Texas) Supt. of Schools, Teague, Tex.

T. G. Ostrom, A.B.(Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn.

C. G. Phipps, Ph.D.(Minnesota) Asso. Prof., Univ. of Florida, Gainesville, Fla.

A. R. Poole, Ph.D.(Calif. Inst. of Tech.) Instr., Univ. of Minnesota, Minneapolis, Minn.

Rev. J. G. Précourt, M.S.(Catholic Univ.) Prof. of Physics, Asso. Prof. of Math., Providence Coll., Providence, R. I.

H. F. Price, Ph.D.(Pennsylvania) Prof., Pacific Univ., Forest Grove, Ore.

E. G. Salisbury, Ph.D. Prof., State Teachers Coll., California, Pa.

H. F. Schroeder, M.S.(Louisiana) Asst. Prof., Louisiana Poly. Inst., Ruston, La.

C. S. Stuckey, A.B.(Hastings) Grad. asst., Univ. of Wyoming, Laramie, Wyo.

W. I. Thompson, A.M.(California) Instr., Los Angeles Jr. Coll., Los Angeles, Calif.

A. W. Tucker, Ph.D.(Princeton) Asso. Prof., Princeton Univ., Princeton, N. J.

Robert Tucker, A.B.(Macalester) Grad. asst., Univ. of Minnesota, Minneapolis, Minn.

J. W. Tukey, M.S.(Brown), A.M.(Princeton) Grad. student, fellow, Princeton Univ., Princeton, N. J.

L. H. Tulloch, A.M.(Brown) Instr., Jr. Coll., San Antonio, Tex.

O. D. Tyner, A.M.(Chicago Law School) Teacher, Chicago Tech. Coll., and Lane Tech. High School, Chicago, Ill.

G. C. Vedova, A.M.(Columbia) Asst. Prof., St. John's Coll., Annapolis, Md.

R. T. Wallace, A.B.(British Columbia) Asst. Prof., Victoria Coll., Victoria, B. C., Canada.

Agnes E. Wells, Ph.D.(Michigan) Prof., Indiana Univ., Bloomington, Ind.

D. E. Whelan, Jr., B.S. in C.E.(Mass. Inst. of Tech.) Dean, Coll. of Science, Loyola Univ. of Los Angeles, Los Angeles, Calif.

Marelena White, M.S.(Louisiana) Asst., Louisiana State Univ., University, La.

P. M. Whitman, A.M.(Harvard) Instr., Harvard Univ., Cambridge, Mass.

It was voted that the invitation to meet at Ohio State University in December 1939 be accepted, and that greetings be sent to the American Mathematical Society on the occasion of their Semicentennial.

W. D. Cairns, *Secretary-Treasurer*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at St. John's University, Collegeville, Minnesota, on Saturday, May 14, 1938. A morning session was held at 11:00 o'clock and was followed by luncheon and an afternoon session at 2:15 o'clock. Professor G. L. Winkelmann of St. John's University presided at the two sessions.

Eighty-one persons attended the meeting including the following thirty-two members of the Association: Mae R. Anderson, C. J. Blackall, L. E. Bush, W. H. Bussey, E. J. Camp, S. Elizabeth Carlson, Sister M. Claudette, R. W. Cowan, Arthur Danzl, J. H. Daoust, Louis De LaSalle, Bernard Dimsdale, Gladys Gibbens, W. L. Hart, H. E. Hartig, J. S. Hickman, Dunham Jackson, A. L. O'Toole, J. M. Rysgaard, R. B. Saunders, M. G. Scherberg, Ole Schey, C. Grace Shover, R. R. Shumway, A. J. Strane, F. J. Taylor, H. P. Thielman, Ella Thorp, A. L. Underhill, K. W. Wegner, Marian A. Wilder, G. L. Winkelmann.

At the business session officers were elected for the coming year as follows: Chairman, W. H. Bussey, University of Minnesota; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: E. J. Camp, Macalester College, A. J. Strane, Duluth Junior College; Louis De LaSalle, St. Mary's College.

It was decided to hold a second meeting this year, a joint meeting with the Mathematics Section of the Minnesota Educational Association in Minneapolis, October 28, 1938.

The following ten papers were presented:

1. "On finding the reciprocal of a matrix" by Dr. K. W. Wegner, University of Minnesota.
2. "Note in factor analysis" by Mary Elveback, University of Minnesota, introduced by Professor Dunham Jackson.
3. "Operator isomorphism and equivalence of ideals" by Dr. C. Grace Shover, Carleton College.
4. "Some present policies in French and German universities" by Professor A. L. Underhill, University of Minnesota.
5. "Note on experimental testing in trigonometry" by Dr. Marian A. Wilder, Research Statistician, University of Minnesota.
6. "Some remarks on modular geometries and their relation to Euler's criterion in the theory of numbers" by Professor W. H. Bussey, University of Minnesota.
7. "Note on the summation of Fourier Series" by Professor Dunham Jackson, University of Minnesota.
8. "Beaming definite integrals" by Professor G. L. Winkelmann, St. John's University.
9. "The utility of indeterminate forms in the first course of the calculus" by Professor M. G. Scherberg, University of Minnesota.
10. "The determination of the characteristic exponent for linear differential equations of the second order" by Dr. R. W. Cowan, College of St. Scholastica, Duluth.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Wegner described a method of finding the reciprocal of a matrix which is based on the Hamilton-Cayley Theorem. The method is not new, but is little known, and it is interesting if not practical.

2. Miss Elveback showed that the observed standard scores (z_{ik}) are transformed into the independent scores or "factors" (γ_{ik}) by the matrix equation $z=A\gamma$ where A is the unitary matrix of the characteristic vectors of the real symmetric correlation matrix $R=(r_{ik})$ and therefore represents a pure rotation. Such a real orthogonal matrix A always exists and the characteristic numbers λ_k are real and equal the variance of the factors γ_k. The condition that a set of numbers be direction cosines of a principal axis of the surfaces of equal frequency was shown to be satisfied by each characteristic vector of R. Thus the

rotation $Z=A\gamma$ rotates the correlation ellipsoids to their principal axes, the equation in terms of the factors and their variances being $\sum_{i=1}^{n}\gamma_i^2/\lambda_i=D$.

3. Dr. Shover proved that ideals which are equivalent in the sense of Dedekind are operator isomorphic and conversely.

5. The types of questions tried were matching, multiple choice, and indirect, none of which were found significantly different from the usual direct questions. Direct questions broken into small units were easier than the entire problem for the class as a whole but harder for the top group. The placement as explained by Dr. Wilder was found useful for predicting success in the course for students having higher algebra in high school.

6. Professor Bussey showed how the solutions of the linear congruence $ax^2+2hxy+by^2+2gx+2fy+c=0$, modulo p, can be pictured by points in a euclidean plane. He referred to such collections of points as second degree "constellations" in finite geometries, modulo p, and showed how their properties lead to a very simple proof of the theorem that the number -1 is a square or a nonsquare, modulo p, according as p is of the form $4k+1$ or $4k+3$.

7. The paper of Professor Jackson called attention to an especially simple proof of the well known fact that if $f(x)$ satisfies a Lipschitz condition the error of the Fejér mean of the partial sums of its Fourier series through terms of the nth order does not exceed a quantity of the order of magnitude of $(\log n)/n$. The error is not greater than a constant multiple of an integral which can be interpreted as giving the value of the corresponding error in the case of the particular function $|x|$, and an upper bound for the latter can be immediately deduced from the explicit form of the Fourier coefficients.

8. Space diagrams, shear diagrams, moment diagrams, slope diagrams, and deflection diagrams for beams are presumed to be developed in the usual manner. Professor Winkelmann developed formulas for the shear, the binding moment, the slope of the beam, and the deflection of the beam at a point. For a cantilever beam with a concentrated load at each end similar formulas were presented as well as for a simple beam with a uniformly distributed load over the entire span.

9. Dr. Scherberg showed by giving several examples that the indeterminate form $\lim_{x\to a}(f(x)-f(a))/(g(x)-g(a))$ appears in a number of places in the first course of the calculus in the process of developing formulas for applications (radius of curvature, *etc.*). Also that these forms are readily evaluated by means of the formula

$$\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\cdot$$

It is believed that the latter method simplifies the development of these formulas and hence also the whole structure of the course.

10. Dr. Cowan showed that the second order differential equation may be transformed to a Riccati equation. Then by requiring that a series consisting of ascending powers of the parameter with undetermined coefficients be a solu-

tion of the Riccati equation, we are led to a set of first order differential equations, the integration of which is easily accomplished. A transformation of variable and integration from zero to $2\pi i$ determine the characteristic exponent within an integer. The solution of the original differential equation for which the characteristic exponent has been determined is shown to converge by setting up an absolutely convergent series which dominates it.

A. L. UNDERHILL, *Secretary*

THE 1938 MEETING OF THE TEXAS SECTION

The 1938 meeting of the Texas Section of the Mathematical Association of America was held at Texas Christian University, Fort Worth, on Friday and Saturday, April 22–23. The meeting was presided over by the chairman, Professor J. H. Binney of the A. and M. College of Texas.

Among the seventy-two persons attending the meeting were the following twenty-three members of the Association: E. F. Beckenbach, J. H. Binney, A. A. Blumberg, H. E. Bray, Myrtle C. Brown, J. E. Burnam, J. G. Chaney, E. L. Dodd, Nat Edmonson, Jr., H. J. Ettlinger, E. H. Hanson, C. M. Howard, A. J. Kempner, H. A. Luther, B. C. Moore, E. D. Mouzon, Jr., M. E. Mullings, G. A. Newton, Maxwell Reade, Ethel A. Rice, C. R. Sherer, Jennie L. Tate, E. V. White.

Local arrangements were in charge of Professor C. R. Sherer, who was assisted by members of the "Parabola," a club composed of those students of Texas Christian University majoring in mathematics. An outstanding feature of the program of entertainment was the dinner on Friday evening for those attending the meeting. Dean E. V. White of Texas State College for Women was the invited speaker at the dinner.

The Section was very glad to have as the guest of honor for this meeting the president of the Association, Professor A. J. Kempner of the University of Colorado. His two addresses were enthusiastically received and were definitely stimulating to the Section membership.

At the business session the following officers were elected for the coming year: Chairman, H. J. Ettlinger, University of Texas; Vice-Chairman, E. D. Mouzon, Jr., Southern Methodist University. At the invitation of Abilene Christian College, Hardin-Simmons University, and McMurray College, all at Abilene, the 1939 meeting was scheduled to meet at Abilene.

The following papers were presented:

1. "The rôle of isomorphism in scientific systems" by Professor A. J. Kempner, University of Colorado.
2. "On continued fractions" by Dr. Walter Leighton, Rice Institute, introduced by Professor Bray.
3. "A highly irregular boundary value problem of order $2n$" by Dr. H. A. Luther, A. and M. College of Texas.
4. "Interior and exterior means obtained by the method of moments" by Professor E. L. Dodd, University of Texas.

5. "A generalization of the theorems of Cauchy and Morera" by Maxwell Reade, Rice Institute.

6. "On the measure of linear point sets" by A. A. Blumberg, A. and M. College of Texas.

7. "On the limit of a sequence of integrals" by J. P. Nash, Rice Institute, introduced by Professor Bray.

8. "Extended expansion theorems" by J. G. Chaney, A. and M. College of Texas.

9. "The Texas Curriculum Revision Movement in the field of mathematics from the viewpoint of the secondary schools" by Bee Grissom, Austin High School, introduced by Professor Ettlinger.

10. "The Texas Curriculum Revision Movement from the college and university viewpoint" by Professor H. J. Ettlinger, University of Texas.

11. "The need for cooperation between high school teachers and college teachers of mathematics" by Professor A. J. Kempner, University of Colorado.

Abstracts of some of these papers follow, numbered as in the list above:

1. Professor Kempner's paper will appear in a later issue of the MONTHLY.

2. Dr. Leighton presented an exposition of continued fractions from a function-theoretic point of view. A general discussion of convergence criteria was given with particular reference to the results of E. B. Van Vleck, A. Pringsheim, O. Szàsz, H. S. Wall, J. Q. Jordan, and the author. Reference was made to the first "gap" theorems for continued fractions, proved recently by W. T. Scott and Walter Leighton.

3. Dr. Luther considered the problem of expanding functions in infinite series whose terms satisfied the equation $d^{2n}u/dx^{2n}+\rho^{2n}u=0$, $n\geqq 2$, and certain highly irregular boundary conditions at the two ends of an interval. It was found that conditions for the expansibility of functions could be derived by finding such conditions for a related problem in which the boundary conditions bore at only one end of the interval.

4. Professor Dodd discussed frequency functions $\alpha^{-1}\Phi(t')$ in x, with reference to estimating by the method of moments the location κ and the scale α associated with x thus: $t'=(x-\kappa)/\alpha$. With k and a as estimates of κ and α, the frequency becomes $y=a^{-1}\Phi(t)$, $t=(x-k)/a$. Now, independent of a and k, $\Phi(t)$ has a mean value μ, and a variance σ^2. For n measurements, X_i, let $\overline{X}$ be the arithmetic mean, and let τ^2 be the variance. By the method of moments, $a=\tau/\sigma$. Thus a is a root-mean-square of $|X_i-\overline{X}|/\sigma$, and $k=\overline{X}-a\mu$, a substitutive mean of X_i. It is indeed an external mean if these measurements are equal numerically, just half of them positive, and the second moment of $\Phi(t)$ is less than twice the square of the first moment. For example, this is the case if $\Phi(t)$ is a Pearson Type III, with origin at the start of the curve, and with the exponent of t positive.

5. Mr. Reade obtained generalizations of the Cauchy and Morera theorems. A typical result is the following: Let $X_j(u, v)$, $j=1, 2, 3$, be a triple of conjugate harmonic functions defined in domain D: (u, v), where D is the interior of the circle $u^2+v^2=1$. Let $X_j(s, t)$, $j=1, 2, 3$, be a conformally equivalent triple de-

fined in $\bar{D}$: (s, t). Then a necessary and sufficient condition that the triple $X_j(u, v)$, $j=1, 2, 3$, map D on a plane is that for all maps of D on domains $\bar{D}$ in the (s, t) plane we have $\sum_{j=1}^{3}[\int_{\bar{c}}\overline{X}_j(s, t)dw]^2=0$, where $w=s+i\cdot t$, and $\bar{c}$ is the map of circle $u^2+v^2=1$ in $\bar{D}$. Further results were obtained by considering the average function defined by

$$A\rho_j(u_0, v_0) = \frac{1}{\pi\rho^2}\int\int_R X_j(u_0 + \rho\cos\phi, v_0 + \rho\sin\phi)dA,$$

where dA is element of area, and R the interior of circle $(u-u_0)^2+(v-v_0)^2=\rho^2$.

6. Mr. Blumberg proved the following theorems: (1) If M and N are bounded linear point sets and $m_i(M\cdot N)=0$, then a necessary and sufficient condition that $m_i(M)+m_i(N)=m_i(M+N)$ is that $H\cdot M$ shall be measurable where H is a measurable subset of $M+N$ such that $m(H)=m_i(M+N)$. (2) If M and N are bounded linear point sets, then a necessary and sufficient condition that $m_e(M)+m_e(N)=m_e(M+N)$ is that for each positive number ϵ there shall exist collections G_ϵ and H_ϵ of mutually exclusive segments, covering M and N respectively, such that $m(G_\epsilon^*\cdot H_\epsilon^*)=0$.

7. Mr. Nash dealt with convergent sequences or families of non-negative summable functions of x, $f(t, x)$ with $\lim f(t, x)=f(x)$, $0\leqq x\leqq 1$, of which the Poisson integral, Fejér sum, *etc.* are typical. It was found that a sufficient condition in order that $f(t, x)$ satisfy the more restrictive condition of Lebesgue is that

$$\int_0^1 g(x)\log\frac{1}{1-x} < \infty,$$

where $g(x)$ is the *monotone* function equivalent to $f(x)$ in the sense that $mE[g(x)<A]=mE[f(x)<A]$. This condition is also necessary if $f(x)$ is itself monotone.

8. Mr. Chaney derived an identity, which he referred to as *the key identity*, which includes Bartky's Identity as a special case. This identity, in conjunction with some fundamental operational formulas, was used to develop some very general extensions of the Heaviside type expansion theorem. The key identity was shown to be the connecting link between the usual scalar type of expansion theorem and the vector-tensor type of expansion theorem. The entire paper was considered from the viewpoint of applications to electric circuit theory.

10. As the University representative of the mathematics section of the Texas State Curriculum Revision Committee, Professor Ettlinger was interested in the following aims: (1) avoiding the dilution or weakening of the high school mathematics courses; (2) strengthening of the fundamentals in these courses; (3) enrichment of essential drill with material taken from everyday experience and from scientific fields; and (4) assurance that those who will pursue studies in scientific work in any field will have adequate preparation to continue their mathematical studies.

11. In this paper Professor Kempner expressed ideas and viewpoints which have already appeared in various papers by him, such as an address printed in this MONTHLY for December 1937.

NAT EDMONSON, JR. *Secretary*

THE FIFTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Mississippi State College, March 11–12, 1938. The chairman of the Section, Professor Dorothy McCoy of Belhaven College, presided.

The attendance was approximately sixty-five, including the following twenty-five members of the Association: T. A. Bickerstaff, H. E. Buchanan, W. E. Cox, G. F. Cramer, D. S. Dearman, W. L. Duren, Jr., Virginia I. Felder, Elizabeth Freas, Deborah May Hickey, Dorothy McCoy, Janet McDonald, A. C. Maddox, B. E. Mitchell, I. C. Nichols, Arthur Ollivier, W. V. Parker, H. L. Quarles, W. D. Reeve, S. T. Sanders, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker.

The following officers were elected for the year 1938–39: Chairman, J. F. Thomson, Tulane University; Vice-Chairman for Louisiana, H. F. Schroeder, Louisiana Polytechnic Institute; Vice-Chairman for Mississippi, Dorothy McCoy, Belhaven College; Secretary, W. V. Parker, Louisiana State University.

The Section was honored by the presence of Professor W. D. Reeve of Teachers College, Columbia University. The value of the meeting was strengthened by the two strong addresses which he gave. In the first he gave a report on the progress made by the Joint Commission on the Place of Mathematics in Secondary Schools. In his address on "Modern curriculum problems in the teaching of mathematics" Professor Reeve considered problems relating (1) to the pupil, (2) to the teacher, (3) to content.

(1) The nature of the high school population has radically changed since 1900. The pupils vary not only in native ability to do high school work, but also in experience and interests. Traditional plans for solving our difficulties have not always been successful; they have had their good points but we need a new philosophy of education to meet modern demands. He considered also the evils of mass education and the injurious effect of emotionalized attitudes.

(2) Higher qualifications, particularly in regard to personality, character, and knowledge of subject matter, are necessary. He spoke also of the reasons for the failure of teachers and of new standards for teacher training.

(3) Professor Reeve emphasized the importance of a six-year course in general mathematics beginning in grade seven and running through grade twelve, as well as certain fundamental reorganization principles.

The following program was given:

1. "A general theory of limits" by Professor H. L. Smith, Louisiana State University.
2. "Homogeneous diophantine equations" by A. A. Aucoin, Louisiana State University, introduced by the Secretary.
3. "On Tschirnhausen transformations" by W. E. Cox, Mississippi State College.

4. "Book values of a specific bond purchase" by Professor I. C. Nichols, Louisiana State University.

5. "Involutions on a complex line" by Professor B. E. Mitchell, Millsaps College.

6. "The differentiability of an arc" by Professor H. T. Fleddermann, Loyola University, introduced by the Secretary.

7. "The N curvatures of a twisted curve in euclidean N-space" by Gordon Walker, Louisiana State University, introduced by Professor H. L. Smith (read by title).

8. "The determination of the Maya unit of measure" by Professor G. F. Cramer, Tulane University.

9. "Functions analogous to trigonometric functions" by Professor V. B. Temple, Louisiana College.

10. "Some notes on divisibility and direct division" by Professor T. A. Bickerstaff, University of Mississippi.

11. "A certain type of correspondence" by E. P. Coleman, Mississippi State College, introduced by Professor C. D. Smith.

12. "The use of operators in evaluating infinite integrals" by Professor J. F. Thomson, Tulane University.

13. "Mean figures" by Bayliss Shanks, Millsaps College, introduced by Professor Mitchell.

14. "A new approach to the solution of the cubic and quartic" by Eckford Cohen, student, Starkville High School, introduced by Professor C. D. Smith.

15. "The human aspect of mathematics teaching" by Professor S. T. Sanders, Louisiana State University.

16. "Expedient compromise between the traditional viewpoint and current trends in mathematical curricula and instructions" by Professor A. C. Maddox, Louisiana State Normal College.

17. "The Joint Commission on the Place of Mathematics in the Secondary Schools" by Professor H. E. Buchanan, Tulane University.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles.

1. In 1922 E. H. Moore and H. L. Smith published in the *American Journal of Mathematics* a general theory of limits, which theory included nearly all the types of limits found in the literature. A notable exception is the approximate limit employed in the advanced work on integration. In the present paper, which has appeared in the *National Mathematics Magazine* (vol. 12, 1938, pp. 371–379), Professor Smith presented a new theory which generalizes the earlier theory and does include the approximate limit as an instance.

2. The Diophantine equation $f(x_1, x_2, \cdots, x_r) = g(y_1, y_2, \cdots, y_s)$, where f and g are homogeneous polynomials of degrees m and n respectively, and m and n are relatively prime, always has an infinitude of integral solutions. Mr. Aucoin showed how these solutions may be obtained in terms of $r+s$ parameters. He further showed that the solutions are general except those for which $f=0$, in which case the trivial solution $x_i = y_j = 0$ is obtained.

5. Employing the method of Laguerre in the representation of the imaginary point Professor Mitchell considered the following problem: Given an involution on a complex line determined by two complex point-pairs, to find its center and its double points, the Jacobian pair. The general result, including the real case of course, is: The initial points of the real representation of the Jacobian pair are doubly cyclic and anticyclic with respect to each pair of initial points of the original pairs, with an identical relation of the terminal point-pairs. The six points involved in each case are the vertices of two hexagons which are inversely similar—the condition of collinearity of the four points involved. The construction involved is that employed by Morley.

6. After defining what is meant by the various orders of differentiability of an arc and a vector function, Professor Fleddermann showed that the sufficient condition for an arc to be nth differentiable at a point is that the vector function which represents it be nth differentiable at that point. He also showed that the necessary and sufficient condition for an arc to be first differentiable at a point is that its contingent, in Bouligand's sense, consist of a single half line or a pair of complementary half lines, and that the necessary and sufficient condition for second order differentiability of an arc at a point is that it have contact of order two at least with some circle at that point.

8. This paper appeared in the June-July 1938 number of this MONTHLY.

9. Professor Temple defined functions, based on the solutions of the differential equation $d^nu/dx^n+u=0$, which were analogous to the trigonometric functions $\sin x$ and $\cos x$. He also showed that functions based on the solutions of the differential equation $d^nu/dx^n-u=0$ could be defined, and are analogous to the hyperbolic functions $\sinh x$ and $\cosh x$.

10. After a review of certain criteria for divisibility by given numbers, in which use was made of appropriate congruences and of sets of remainders resulting from dividing powers of ten by the given numbers, attention was given by Professor Bickerstaff to the question of the multiples of divisor removed by casting out processes.

11. Mr. Coleman considered the relation $t^2+yt+x=0$, where x and y are the usual rectangular coördinates of a point in the plane and t refers to a point on a line. For points in the plane such that $y^2-4x=0$ there exists a (1–1) correspondence between the points of the parabola and the points of the line. The following theorems were proved: (1) The arbitrary choice of a point $P(x, y)$ in the plane fixes points t_1 and t_2 on the line, and their correspondences Q_1 and Q_2 on the parabola. The point P is the pole of the line Q_1Q_2. (2) If two points P_1 and P_2 be chosen in the plane so that the corresponding points on the line are harmonic, then P_1 and P_2 are conjugate points.

12. The relationship between the impedance operator of Heaviside $1/z(p)$ and the indicial admittance function $A(t)$ was shown in several cases. The infinite integral theorem $1/az(t)=\int_0^\infty e^{-ax}A(x)dx$ was then introduced, and this theorem used to evaluate several infinite integrals.

13. Extending the mean division of a line segment arithmetically, geometri-

cally, and harmonically to other geometric figures, Mr. Shanks divided a square by means of parabolas into the same three means. Then employing the right conoid as a sort of general mean figure between a right circular cone and a right circular cylinder, he secured a geometric mean section by a secant plane, and further showed that the conoid is the harmonic mean between the cone and the cylinder in respect to the volume, centroid, and moment of inertia referred to its axis.

14. Mr. Cohen's method was based upon equivalence of forms expressed as combinations of linear and quadratic factors. Each solution was obtained by a resolvent equation, the solution of which leads directly to evaluation of the coefficients in the factors.

W. V. Parker, *Secretary*

THE TWENTY-FOURTH ANNUAL MEETING OF THE KANSAS SECTION

The twenty-fourth annual meeting of the Kansas Section of the Mathematical Association of America was held in conjunction with the Kansas Association of Mathematics Teachers at the State Teachers College, Pittsburg, on Saturday, April 2, 1938. This was the first meeting of the two organizations to be held in Pittsburg, and there was much appreciation expressed over the cordial welcome of President Brandenburg of the College, the delightful luncheon in the College dining room, and the evidences of careful planning for our comfort and pleasure by the department, under the leadership of Professor J. A. G. Shirk.

The attendance was one hundred and fourteen, including the following twenty-nine members of the Association: Sister Mary N. Arnoldy, R. W. Babcock, Wealthy Babcock, Lois E. Bell, Florence L. Black, R. D. Daugherty, Lucy T. Dougherty, D. D. Driver, W. H. Garrett, A. J. Hoare, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza A. Mossman, O. J. Peterson, G. B. Price, C. B. Read, B. L. Remick, J. A. G. Shirk, G. W. Smith, R. G. Smith, E. B. Stouffer, W. T. Stratton, C. B. Tucker, J. J. Wheeler.

In the morning there was a joint session of the two organizations, which was presided over by Professor C. B. Tucker of State Teachers College, Emporia. The program included papers by Professor A. R. Congdon of University of Nebraska, "Mathematical requirements for high school graduation and college entrance"; and by Professor R. W. Babcock of Kansas State College, Manhattan, "Scientific surplus"; also a report by Professor U. G. Mitchell, University of Kansas, on "The work of the Joint Commission on the Place of Mathematics in Secondary Schools." Professor Anna Marm, Bethany College, Lindsborg, was president of the Kansas Association, and arranged the program.

Professor C. B. Tucker, vice-chairman of the Section, presided at the meeting of the Section after the luncheon, the chairman, Professor W. G. Warnock, being on leave of absence for this semester. The officers chosen for the coming year were: Chairman, C. B. Tucker, Kansas State Teachers College, Emporia; Vice-Chairman, C. B. Read, University of Wichita; Secretary, Lucy T. Dougherty, Junior College, Kansas City.

The following program was presented:

1. "On Bernoulli polynomials and numbers" by Dr. G. C. Munro, Kansas State College, Manhattan, introduced by Professor Stratton.

2. "Vexing minor problems of the mathematics curriculum" by Professor C. B. Read, University of Wichita.

3. "The Pascal configuration in a modular geometry" by C. E. Rickart, University of Kansas, introduced by Professor Mitchell.

4. "A program for the Association" by Professor G. B. Price, University of Kansas.

Abstracts of the papers follow:

1. Three different sets of definitions for the numbers and the polynomials of Bernoulli were shown by Dr. Munro to be equivalent.

2. Professor Read called attention to problems common to Kansas colleges and universities. Among such we find the student deficient in the usual college entrance mathematical subjects, the content of certain college courses, the use of textbooks. A temporary solution was given to the problems arising when the teaching staff is not large enough to handle the load.

3. Following a brief outline of some of the properties of the Pascal configuration Mr. Rickart examined them in the modular plane $PG(2.5)$ and showed that every Pascal configuration reduced to a Desargues configuration which is self-polar as to the fundamental conic. The 15 Salmon points, the 20 Steiner points, 30 of the 45 Pascal points, and the 60 Kirkman points reduce to the 10 points of the Desargues configuration, while the 15 Steiner-Plücker lines, the 20 Cayley-Salmon lines, and the 60 Pascal lines reduce to the 10 lines. The (1,1) correspondence between the above points and lines becomes a correspondence between poles and polars with respect to the fundamental conic.

4. This paper appeared in the October 1938 number of this MONTHLY.

LUCY T. DOUGHERTY, *Secretary*

GEOMETRIES*

MAYME I. LOGSDON, University of Chicago

When in high school you became acquainted with euclidean geometry it seemed completely in accord with your world of experience. When later you learned of the existence of other geometries, it is quite likely that your instinctive reaction was the thought that these were probably creations of the pure mathematician for which there could be no practical use since we live in a euclidean space.

When a physical or geometrical theory has been for a long time used to explain certain observed data, it is not surprising that there is a tendency to believe that this is perforce the correct theory and to expect that henceforward nature will act in the manner predicted by the theory. The danger inherent in this attitude is that the mind is closed against alternative theories which indeed may provide an equally good interpretation of the same data.

Not until the investigator realizes that no single theory has been found or will be found which explains completely the behavior of Nature but that there are numerous theories which explain most of the facts of observation, will he be equipped to study with interest and profit the relation of fact to theory. Only then can he be expected not to make the error of concluding that other situations, not within the range of observation, are necessarily thus and so because his favorite theory so predicts it.

From the year 1826 when Lobatchevsky, to quote Einstein's phrase, *challenged an axiom*, to the present day, much has been heard of the non-euclidean geometries and the Riemannian geometry. Because the latter is the geometry of the general relativity theory, it challenges the interest of all who are trying, sometimes more and sometimes less casually, to understand the developments and implications of this theory.

In order to understand the fundamental properties of Riemann space, let us start with a brief description of the structure of an axiomatic geometry and follow with a discussion of the structure of an analytic geometry. With this preliminary it will be possible to proceed toward the goal which has been set.

Three-dimensional geometry as developed by Euclid is an axiomatic geometry. It starts with three undefined notions called points, lines, and planes for which are assumed certain relations, called postulates or axioms. We need only recall a few of the (according to Hilbert) twenty axioms, *e.g.*, through two distinct points one and only one line can be drawn, a line contains at least two points, *etc.* On the basis of these axioms and his definitions, Euclid built a geometry. An axiomatic geometry is defined as *a set of undefined terms, of axioms and definitions, and their consequences.*

On the other hand *analytic geometry of three-space is a purely arithmetical theory.* We define a point as a triple of numbers (x, y, z) and say that the totality of such triples constitutes a three-dimensional space. In addition we must agree upon a *distance function*, *i.e.*, we *define* as the distance between two points $P(x, y, z)$ and $P'(x', y', z')$ the *number* obtained by substituting the coördinates of the two points in a chosen formula. Finally we give definitions of plane and line. Our analytical geometry is the logical consequence of this set-up.

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* Presented for the Slaught Memorial Volume. Read at a meeting of the National Council of Teachers of Mathematics in Atlantic City, February, 1938.

You will note that there are no axioms and no undefined terms, but that we can get many analytical geometries by choosing different distance functions, and if we so desire, different definitions of line and plane. As a particular instance of an analytical geometry of three-space let us choose the following as definitions:

(a) The distance PP' is the number $\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$.

(b) A plane is the totality of points (triples) which satisfy a given linear equation $ax+by+cz+d=0$ where a, b, c, and d (not all zero) are given except for a proportionality factor.

(c) A line is the totality of points (triples) which satisfy two such equations, $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$, where the continued proportion $a:b:c=a':b':c'$ does not hold.

With these agreements it is possible by use of purely algebraic methods to prove all of the axioms of Euclid. And conversely, if we start with our axiomatic geometry, we can prove that points, lines, and planes, hitherto undefined, can be designated by triples of numbers and linear equations. Hence the properties of euclidean three-space are equally well investigated by use of Euclid's axiomatic geometry or by the arithmetical methods of analytical geometry. Recalling our definition of a geometry, we may say that the euclidean geometry of a three-space S_3 is nothing more nor less than the logical consequences of the twenty axioms and that it is nothing more nor less than the logical consequences of defining a point as a triple of numbers and adding the definitions of distance, plane, and line as given above in (a), (b), and (c).

The last statement is equivalent to this: *The two geometries are logically equivalent*, or it is even better to state it thus: *They are two logical representations of one and the same logical system.*

You are doubtless thinking that heuristically the concepts of both structures were motivated by the facts of experience. Indeed the development of analytical geometry by Descartes and Fermat had its origin in the fact that when referred to a rectangular system of coördinates, a point is completely located by three numbers, its coördinates, and that the distance formula comes directly from the theorem of Pythagoras.

This is indeed true but it does not invalidate the contention that an analytical geometry which is logically equivalent to Euclid's geometry may be looked upon as a purely mathematical creation and all of its theorems and their consequences developed without at any time making an application to the world of experience.

Indeed you spare yourself uncomfortable moments if you look upon Euclid's axiomatic geometry as a purely mathematical creation; it is not a simple matter to attempt to explain any part of it in the world of experience. For in this world there is no point or line or plane, and if we are willing to be guided by our intuition as regards these terms, we may well be puzzled at the necessity for certain of the axioms, for example, according to our intuition surely two distinct points can be joined by one and only one line. Why then did Euclid find it necessary to include this among his axioms? It will take only a cursory examination to convince you of the beauty and perfection of Euclid's axiomatic geometry as a purely mathematical structure.

As we proceed to examine other geometries let us agree first of all to look upon them as logical structures and not let our minds be puzzled as to the possible existence of spaces with the properties described.

You know the history of the famous parallel axiom of Euclid. Given a line g and a point P not on the line, there exists in the plane determined by P and g one and only one line through P which does not intersect g. For approximately twenty centuries mathematicians believed that this statement (you note that it is completely satisfying to your geometric intuition) was capable of being *proved* by use of the remaining axioms but no effort to produce a satisfactory proof was successful. It was Lobatchevsky who first conceived the idea of investigating the logical consequences of postulating all of the relations given in Euclid's axioms except the parallel axiom, for which he substituted one reading "an infinite number of lines can be drawn through P having no point in common with g."

On this foundation he and Bolyai (independently) constructed a geometry which of necessity differs widely from Euclid's geometry. Three of these differences, needed in the sequel, will be stated here. You recall that in Euclid's geometry one of the undefined terms is *line* and there is a theorem that a straight line is the shortest distance between two points. In this non-euclidean geometry the shortest path joining two points, called a *geodesic*, is not a straight line (in the euclidean sense). Furthermore, in a triangle, a figure whose perimeter consists of three geodesics, the sum of the three angles is 180° in Euclid's geometry while in the non-euclidean geometry under discussion in no figure whose perimeter consists of three geodesics is the angle sum exactly 180°; in fact, it is *always less than* 180° differing from this number by very little when the triangle is small but differing more and more as the area of the triangle increases.

Still another notable difference in the two geometries has regard to similar figures. In the new geometry two figures cannot be similar when they have not the same areas.

We may summarize the last paragraphs by formulating a reply to the question, What is a non-euclidean three-dimensional space? The answer is, a non-euclidean three-dimensional space is a system of things called points, lines, and planes with relations which satisfy all of the euclidean axioms except the parallel-axiom which is replaced by a contrary one. Non-euclidean geometry is a set of all logical consequences of this system of axioms.

Whether or not these non-euclidean axioms conform to the so-called space of observation is of no consequence to the development of the deductive system of non-euclidean geometries.

One question concerning the non-euclidean geometry is certainly pertinent. The novelty of the theorems when compared with the euclidean theorems, may raise doubts in the mind as to whether it may not possibly contain contradictions, *i.e.*, whether from the axioms may not be obtained a theorem which negates another theorem already established. The reply, says Nöbeling,* is that there is just as little likelihood for this in non-euclidean as in euclidean geometry. Indeed Klein succeeded in constructing *within the euclidean three-space* a set of things designated as points, lines, and planes which do not satisfy all of the euclidean axioms.

The non-euclidean geometry which has been described is the so-called *hyperbolic* geometry. A second non-euclidean geometry, called *elliptic* geometry, has been constructed by replacing the parallel axiom by one asserting that through P no line can be drawn which does not meet g. We need not further study the details of these geometries but we raise the significant question: *Are there analytical geometries which are logically equivalent to these geometries?* The answer is affirmative but before they can be described it is desirable to review briefly some of the distinguishing features of a branch of the analytical geometry of euclidean space. This is called *differential geometry* and is used for the study of curves and curved surfaces in euclidean space. It provides methods for the computation of lengths of curves lying on a surface, of angles between such curves, of areas, and other metrical notions. Of especial interest are the geodesic lines on a surface. On a sphere they are arcs of great circles, but on the surface of the earth, for example (not a perfect sphere), if A and B are two points separated by a mountain, the geodesic joining A and B might go around the mountain. The angle sum of a figure made up of three geodesics differs from 180° only if the surface is curved. In this differential geometry a smooth surface F is defined as the totality of all points (triples), (x, y, z), which satisfy a relation

$$F: \qquad f(x, y, z) = 0,$$

where the function f is differentiable as many times as needed with respect to each of the three variables. Such a surface has a tangent plane at every point and the surface can be represented in the neighborhood of a point (stückweise, as the Germans say) by two coördinates, say u and v:

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

Gauss found that when a piece of a surface is described by two coördinates as above, to each of its points (u, v) can be associated three functions

$$g_{11}(u, v), \qquad g_{12}(u, v), \qquad g_{22}(u, v),$$

whose values, in general, change when the point changes from a given (u, v) to any other (u', v').

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* Krise und Neuaufbau in den exakten Wissenschaften, Deutike, Leipzig und Wien, 1933, p. 77.

These functions serve for finding length of arc, angle magnitude, and area. In particular, for two points sufficiently near each other the square of the length of the geodesic arc differs very little from

$$g_{11} \cdot (u - u')^2 + 2g_{12} \cdot (u - u')(v - v') + g_{22} \cdot (v - v')^2.$$

When the equation $f=0$ is linear, so that its locus is a plane, if we choose for u, v the rectangular cartesian coördinates x, y we find that $g_{11}=g_{22}=1$ and $g_{12}=0$, thus giving as the square of the length of the geodesic arc the number

$$(x - x')^2 + (y - y')^2.$$

By use of the g-function we obtain a function $k(u, v)$ whose value for a given point is called the *curvature* of the surface at that point. The value of k differs from zero according as the angle sum of the triangle formed by three geodesic lines differs from 180°. We note the special cases:

1) When F is a plane, $k=0$.

2) When F is a sphere with radius r, $k=1/r^2$, *i.e.*, k is constant and *positive.*

3) If F is a surface of *everywhere positive curvature*, the angle sum of any geodesic triangle of F is *greater than* 180°.

4) If F is a surface of *everywhere negative curvature*, the angle sum is *less than* 180°.

Riemann extended these considerations of differential geometry to higher dimensions and it is precisely this extension which gives us the space which we name Riemann space. In R_4 lengths of arcs, angle magnitudes, areas, and volumes of pieces of a three-dimensional solid M can be computed by the use of six functions g_{11}, g_{22}, g_{33}, g_{12}, g_{13}, g_{23}, functions of three coördinates.

Through an arbitrary point P we can construct a surface which contains the geodesic lines through P of our manifold M. This surface F has a curvature at P. If the manifold M has the property that this curvature is constant as the point P varies on M and its geodesic surface F varies with P, the Riemann manifold (or Riemann space) M is said to be a space of constant curvature k. In particular, if M is a euclidean S_3, $k=0$, since the geodesic surfaces are planes. Also Riemann spaces with constant negative (positive) curvature are hyperbolic (elliptic), *i.e.*, the geodesic lines all satisfy the axioms of hyperbolic (elliptic) geometry.

We can call a Riemann space with constant positive curvature a spherical space. It is the three-dimensional analog of the surface of a sphere. On it, two geodesic lines have *at most one* common point. These Riemann spaces of constant curvature constitute only a narrow special case of the general notion of Riemann space. In general, we call a Riemann space *curved* if triangles formed of geodesic lines have angle sums different from 180°.

To summarize, then, in analytical geometry a four-space is the totality of all quadruples of numbers. A 3-dimensional Riemann space (or Riemann manifold) contained within this space is a triple-infinitude of these points; in particular it may be the totality of points which satisfy a single equation

$$f(x, y, z, t) = 0,$$

with suitable differentiability properties. It can be described in pieces by three coördinates and for it we have given *six* functions of these coördinates which may be used to compute lengths, angle magnitudes, areas, volumes, *etc.*

In an analogous manner a Riemann n-space in a euclidean space of $N>n$ dimensions is defined. The geometry of a Riemann space of whatever dimension is merely the logical consequences of this set-up. In particular, a change in one or more of the g-functions makes a change in the geometry.

Let us now return to a consideration of the relation of these logical systems to our world of experience. In this we have serious handicaps some of which have already been mentioned. If we are to apply our geometries to observations you may make your own choice as to what you are going to look upon as a point, but if we need a straight line of considerable length, perhaps a light ray will be the best choice. We cannot, of course, *observe* any thing which will be analogous to a three-dimensional manifold immersed in a four-space though we know that an event needs a quadruple of numbers for its description, three for its space location and one for the time.

We cannot even derive comfort by attempting to generalize the experiences of a flat-lander whose home is on a plane immersed in a three-space because, granting that the flat-lander does not know that there is a third dimension and consequently does not know any of its properties, *we* have this knowledge and can therefore interpret the things which to him are mysterious.

Let us pursue this a bit further. Imagine a rectangular Cartesian reference frame, say in your home town, and let the flat-landers be our shadows on the ground, $z=0$. Their homes are the shadows of our homes, rectangles or polygons. If a small object is within a certain one of these rectangles, it is not possible for the flat-lander to remove it to a position outside of the rectangle without tearing out a section of the (one-dimensional) wall. But you, a three-dimensional creature, can pick up the object from the plane $z=0$, lift it to the plane say $z=1$, move it in this plane to a suitably chosen new position, then lower it again to $z=0$. As a matter of observation of the flat-lander, the object suddenly disappears from sight and as suddenly and mysteriously reappears in another position.

These motions can be explained quite simply in the language of the analytical geometry of three-space.

A second example is not so simple. Our flat-lander has a glove on his right hand, the shadow of your right-hand glove. He cannot put it on his left hand. If you rotate this glove 180° about an axis through the wrist and any finger, it is now a perfect fit for his left hand though it is certainly not congruent to your left-hand glove. All of these motions can be explained by the analytical geometry of three-space, but the explanation is purely analytical, *i.e.*, arithmetical and algebraic, not physical.

If Sir Oliver Lodge or any of the believers in spiritualism could convince us that our space is immersed in a physical four-space, it would at least divert us to try to imagine physical explanations for many of the phenomena which are clearly explained algebraically by the analytical and differential geometries of four-space. For example, a chain consisting of two or more solid links cannot be unlinked in three-space without breaking or filing some of the links. A string of finite length in which a simple knot has been tied after which the ends of the string have been fastened together, cannot be unknotted in three space without cutting the string or separating the ends. In the analytical geometry of four-space, the chain can be unlinked without severing any link and the string can be unknotted without separating the ends provided that the chain and the string were actually contained in a three-space within the four-space.

Since there is absolutely no evidence of a four-dimensional physical space, why not be content to accept an analytical explanation of a fact not within the possibility of observation as an *analytical* explanation and not attempt to find a physical explanation?

On the other hand we are certainly justified in seeing how far we can go in fitting observed data to our geometries provided that we agree to be open minded and examine our data in their relation to each of our geometries in turn.

One of the distinguishing features of the euclidean and non-euclidean geometries was found to be the angle sum. If you draw a triangle with a ruler on a sheet of paper and measure the angles with a protractor, while you do not obtain precisely 180° for the angle sum you certainly obtain this number within reasonable limits of error. But in hyperbolic geometry we found that the angle sum differs from 180° very little for small triangles. We raise the question, how large must the triangle be before we can detect a measurable difference between the angle sum and 180°? This question occurred to Gauss (1777–1855) who sought an answer by choosing three mountain peaks as the vertices of his triangle and light rays as the sides. His result was 180° within reasonable limits of error, again an inconclusive result since, while he did not prove that euclidean geometry does *not* hold neither does his result justify the conclusion that hyperbolic geometry does not hold. The triangle which he used must surely be called a small triangle when compared with one whose vertices are fixed stars.

It appears that after this experiment of Gauss, the question was shelved until it became of importance in the general theory of relativity, a theory in which it is assumed that the world of experience is a three-dimensional Riemann space.

We have noted that the properties of such a space depend on the choice of the g_{ik}-functions and you would expect that these functions in the present case would be chosen in such a manner that with a given aggregation of mass points (subject only to the law of gravitation), the geodesic lines would be light rays. A complication arises from the fact that the masses in our universe do not form a static system. Suppose that the functions g_{ik} have certain values in the neighborhood of a point P today but that tomorrow a heavy mass has come quite near to P. The values of g_{ik} cannot be constant but must depend on the time in such a manner that the formulas in use do not fail to determine the geodesics.

Analytically, as already stated, the Riemann space is assumed to be a three-dimensional totality of points, *quadruples*, contained within the totality of all quadruples, *i.e.*, contained within a four-dimensional euclidean space whose general point (quadruple) is given by (x, y, z, t), which in the physical application consist of three position coördinates and one time coördinate.

Moreover, since the geodesic lines and light rays coincide and since the geodesic lines are affected by a change in the distribution of the mass, it follows that light rays are affected by a change in the distribution of mass; in other words, a light ray from P to P' would not describe the same path today and tomorrow if in the meantime a heavy mass had moved close to a point of today's path.

This is a prediction from the differential geometry capable of verification by observation and fortunately for the proponents of the theory they did not have to wait long for an opportunity to test it. At the time of a solar eclipse it was definitely confirmed by observation both quantitatively and qualitatively that the light ray sent out by a fixed star was bent in the force field of the sun. An important conclusion from this is that in our universe there are triangles whose sides are light rays but whose angle sum is not 180°. Hence we conclude that the space of experience is curved. Furthermore, the difference between 180° and the angle sum of a triangle with given vertices is not constant but varies as large masses approach or recede from its sides. Hence in the world of experience, not only does the euclidean geometry not hold, but neither does the hyperbolic nor the elliptic.

A conscientious objector now comes forward with the question: Is it not possible to make an equally plausible explanation of such phenomena by assuming that the space is truly euclidean but that in this space light rays are not straight? Indeed this is a possibility which has been studied. The consequence of this assumption is that since light rays do not move along the shortest lines but along highly complicated curves, we obtain an extremely complicated theoretical physics. Further, we should be left in the world of experience (assumed euclidean) with no analog of a geometrical line, an entity which plays a part of much importance in geometry. Thus we can find in our world of experience absolutely no observed illustrations of the relations given by the twenty axioms. Would it not seem paradoxical to call such a space euclidean?

In contradistinction to this situation, we find that the logical consequence of assuming that our space of observation is a Riemann space is a set of simple and elegant formulas, the formulas of the relativity theory. Since we have agreed in advance that we are not going to permit ourselves to fall into the error of accepting *any* theory as a *correct* theory in the sense that its conclusions are to be accepted without question for situations not capable of observation, it is surely reasonable that as far as it is valid we choose the theory which provides the simplest formulas and the easiest explanations.

There is another phase of the relativity theory which is a stumbling block for the uninitiated. This is the assertion that the Riemann space is closed, *i.e.*, unbounded (unbegrenzt) and finite. What is meant by the statement that a space is closed? This is a term whose definition comes to us from the youngest branch of geometry, topology. I shall first give you an example of a closed surface (two-dimensional) and of a surface which is not closed, this for the sake of your intuition, and shall then formulate a precise definition which may be generalized for spaces of arbitrary dimension.

The surface of a sphere is a closed surface, it has no boundary (is unbounded) and is finite since the distance between two arbitrary points (measured along a geodesic) does not exceed a certain fixed number. A circular disk is not closed. It is finite but it has a boundary.

Definition. A surface is called *closed* if it can be cut into a finite number of pieces in such a manner that each bounding arc belongs to exactly two pieces and if the distance between two arbitrary points does not exceed a certain fixed number.

Since after a closed surface has been cut in this manner there is no *free* bounding arc (*i.e.*, no bounding arc which is found on only one piece), the closed surface is called unbegrenzt. We may equally apply the term "closed surface" to a surface made up of a finite number of pieces provided that we can find a scheme for associating the bounding arcs in advance so that each will belong to exactly two pieces.

In the light of this definition let us examine the euclidean plane. It can indeed be cut into pieces in such a manner that every bounding edge belongs to exactly two pieces but there will not be a finite number of these pieces if each piece is to have the property that the distance between two arbitrary points has an upper bound. Hence the euclidean plane is not a closed surface.

The same definition with appropriate change in wording applies to a three-dimensional space. It is closed if it can be cut into a finite number of pieces (of three-space) in such a way that every bounding surface appears on exactly two space pieces. If it is closed, it is unbegrenzt and finite, but of course not conversely.

The relativity theory assumes that the astronomical space of experience is closed. Since each of the finite number of pieces of this space has a finite diameter (*i.e.*, two arbitrary points have a finite distance), it follows that the same is true of two arbitrary points in the space and, consequently, the number of stars is finite, and the total mass is finite.

Again the conscientious objector comes forward with the question: Since you assert that space is finite, should it not be possible to come to the end? He is forgetting that the space is *closed*. Let us transfer our two-dimensional flat-lander to the surface of a sphere and see what experiences might befall him which are not possible on the open euclidean plane. Let him draw a small triangle with arcs of great circles as sides (they appear to him to be straight lines on the surface, since they are short). He finds that the angle sum is approximately 180° and thinks that he is living on a two-dimensional euclidean plane, provided that the flat-lander is very small as compared with the surface of the sphere.

But suppose that these flat-landers are venturesome and go on extensive exploring trips. If they measure large triangles they will find that they are dwelling on a curved surface because the angle sum of a large triangle is greater than 180°. Suppose that they build telescopes—two dimensional ones, of course. All light rays entering the telescopes will come from two-dimensional objects on the sphere and will travel from the object to the lens on a geodesic. Suppose that they succeed in building a telescope sufficiently refined to bring in a light ray from the most distant object in their world. Our flat-lander is amazed to find that he is looking at the other end of himself.

Do you wish to extend the analogy and ask if such a thing is possible for you? Could you imagine a telescope sufficiently powerful to bring to you a view of the other side of the earth, a view carried by a light wave completely around our universe? The answer to that is that the estimated diameter of our universe is 150 milliards of light years so that if you see the other side of the earth you will see it as it was 150 milliards of light years ago.

And now for fear that the preceding paragraphs have made you forget some earlier remarks I shall quote some remarks by G. A. Bliss published in 1933 in the AMERICAN MATHEMATICAL MONTHLY and reprinted in *A Mathematician Explains.*

> The theory of general relativity as applied to the solar system is already threatened, even before its usefulness has been completely established. Recently, when Einstein and DeSitter were sojourning in the United States, the newspapers reported them as discussing still newer theories from which it might be concluded that the universe is finite. This conclusion should never be made. All that we could be justified in saying is that the data which we have concerning the distances and motions of the stars are in closer accord with a theory which is finite than with others which have not this property. The finiteness which can be demonstrated is a property of the theory and not of the universe.

If it has amused you to have the imagined experiences of the flat-lander used to demonstrate the properties of curvature, of being curved and of finiteness, you might wonder what would happen to him if he were placed on a surface whose properties are topologically different. Let us move him to a Möbius band, and let us assume that he is a right-handed creature and wears his heart on the left side just as we do. As stated earlier, as long as he stays near home and measures small triangles, he may well think that he dwells on a euclidean plane and that euclidean geometry holds in his world. But when he ventures farther afield, he will soon learn that angle sums are not always 180°. The amazing experience however will come to him who ventures to walk, say that he remains constantly within sight of the edge of his world and keeps it always on his right. If he travels long enough he will return to his home and family, but what a return. To those who have remained at home, he appears to be left-handed and to wear his heart on the right side. He himself knows full well that he has undergone no anatomical change and it seems to him that all of the stay-at-homes are now left-handed and have their hearts on the right side.

We have an analogy to this on the earth. The man who goes around the world without living through one day twice or else omitting a day can have quite an argument with those who stayed at home about the calendar. We avoid all such trouble by establishing an International Date Line whose use you understand. On the Möbius band, uniformity would be attained if at some point on the band a line should be drawn across the band and if it were decreed that any person crossing this line should from that moment interchange the words "right" and "left."

THE PERIODIC FUNCTION OBTAINED BY REPEATED ACCUMULATION OF A STATISTICAL SERIES

E. J. MOULTON, Northwestern University

1. Introduction. It has been noted as a remarkable fact that in treating a random statistical series by a certain iterative process a periodic function results.* This conclusion was based on calculations with certain series which were assumed to be random. In the calculations discrete series were used but the conclusion was stated for the case of a function of a continuous variable.

It is the object of the present paper to show why this periodic function is obtained. For simplicity the case of a continuous variable will be treated, so that the theory of Fourier series is available. It will be shown that the iterative process leads to a multiple of the first non-vanishing term of a Fourier series development of the function. It will be found that the process has no special significance for a *random* series, leading, as it does, to similar results for a wider class of functions. The conclusion appears more as a characterization of the iterative process than of the statistical series. To show that the iterative process is particularly valuable one would need to show that it gives the first term of the Fourier series more readily than do other methods. Its practical significance in an application of statistics to economics seems doubtful since the periodic function obtained depends so largely on the choice of the interval over which the data extend.

2. The iterative process. Let $x(s)$ be a function (statistical series) defined on the interval $-a \leqq s \leqq a$. We assume that the function (series) is referred to its arithmetic mean, that is, that

$$\int_{-a}^{a} x(s)ds = 0.$$

Instead of assuming that $x(s)$ is a random series we assume that it is representable by a Fourier series which converges at every point of the interval $(-a, a)$:

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* First pointed out by Dr. Herbert E. Jones in a paper entitled, "The theory of runs applied to time series," presented at the Third Annual Research Conference of the Cowles Commission. Noted by H. T. Davis, Mathematical adventures in social science, this MONTHLY, vol. 45, 1938, p. 102; see note on page 105 by the present writer.

$$
\begin{aligned}
x(s) &= A_1 \cos \frac{\pi s}{a} + A_2 \cos \frac{2\pi s}{a} + \cdots \\
&\quad + B_1 \sin \frac{\pi s}{a} + B_2 \sin \frac{2\pi s}{a} + \cdots \qquad (1)\\
&= C_1 \cos \left(\frac{\pi s}{a} - \alpha_1\right) + C_2 \cos \left(\frac{2\pi s}{a} - \alpha_2\right) + \cdots ,
\end{aligned}
$$

where for $j=1, 2, 3, \cdots$,

$$C_j = \sqrt{A_j^2 + B_j^2}, \quad \cos \alpha_j = A_j/C_j, \quad \sin \alpha_j = B_j/C_j.$$

By definition the *first accumulation* of $x(s)$ on $(-a, a)$ is

$$y_1(s) = \int_{-a}^{s} x(s)ds - \frac{1}{2a}\int_{-a}^{a}\int_{-a}^{t} x(s)dsdt = S(x(s)), \tag{2}$$

where S is the linear operator

$$S = \int_{-a}^{s} - \frac{1}{2a}\int_{-a}^{a} dt \int_{-a}^{t} .$$

We observe that $y_1(s)$ is referred to its arithmetic mean, since

$$\int_{-a}^{a} y_1(s)ds = 0.$$

We now form a sequence of accumulations $y_1(s)$, $y_2(s)$, $\cdots$ by repeated application of the operator S, which we call *the iterative process*:

$$y_{k+1}(s) = S(y_k(s)), \qquad k = 1, 2, 3, \cdots . \tag{3}$$

3. The conjecture and the theorem. It has been conjectured by others that when $x(s)$ is a random series

$$y_k(s) \sim A \cos \left(\frac{\pi s}{a} + \alpha_n\right),$$

that is, that $y_k(s)$ approaches a cosine form with period $2a$ when k is increased indefinitely. We shall prove the following

THEOREM. *If $x(s)$ is a function representable by a Fourier series convergent at every point of the interval $(-a, a)$ the first non-vanishing term of which is $C_n \cos (n\pi s/a - \alpha_n)$, then*

$$y_k(s) \sim A \cos \left(\frac{n\pi s}{a} - \alpha\right),$$

that is, the accumulations approach a cosine form with period $2a/n$ when k is increased indefinitely.

4. Proof of the theorem. One readily finds from the definition of the operator S that for any positive integer q

$$\begin{aligned} S\left(C_q \cos\left(\frac{q\pi s}{a} - \alpha_q\right)\right) &= \frac{a}{q\pi} C_q \sin\left(\frac{q\pi s}{a} - \alpha_q\right) \\ &= \frac{a}{q\pi} C_q \cos\left(\frac{q\pi s}{a} - \left(\alpha_q - \frac{\pi}{2}\right)\right). \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned} y_1(s) = \frac{a}{\pi} C_1 \cos\left(\frac{\pi s}{a} - \left(\alpha_1 - \frac{\pi}{2}\right)\right) &+ \frac{a}{2\pi} C_2 \cos\left(\frac{2\pi s}{a} - \left(\alpha_2 - \frac{\pi}{2}\right)\right) \\ &+ \frac{a}{3\pi} C_3 \cos\left(\frac{3\pi s}{a} - \left(\alpha_3 - \frac{\pi}{2}\right)\right) + \cdots, \end{aligned}$$

and for $k = 1, 2, 3, \cdots$

$$\begin{aligned} y_k(s) &= \left(\frac{a}{\pi}\right)^k C_1 \cos\left(\frac{\pi s}{a} - \left(\alpha_1 - \frac{k\pi}{2}\right)\right) + \left(\frac{a}{2\pi}\right)^k C_2 \cos\left(\frac{2\pi s}{a} - \left(\alpha_2 - \frac{k\pi}{2}\right)\right) \\ &\quad + \left(\frac{a}{3\pi}\right)^k C_3 \cos\left(\frac{3\pi s}{a} - \left(\alpha_3 - \frac{k\pi}{2}\right)\right) + \cdots \\ &= \left(\frac{a}{\pi}\right)^k \left\{C_1 \cos\left(\frac{\pi s}{a} - \alpha_{1k}\right) + \frac{1}{2^k} C_2 \cos\left(\frac{2\pi s}{a} - \alpha_{2k}\right)\right. \\ &\quad \left. + \frac{1}{3^k} C_3 \cos\left(\frac{3\pi s}{a} - \alpha_{3k}\right) + \cdots \right\}, \end{aligned} \tag{5}$$

where

$$\alpha_{1k} = \alpha_1 - \frac{k\pi}{2}, \quad \alpha_{2k} = \alpha_2 - \frac{k\pi}{2}, \cdots .$$

It seems fairly obvious that the factors $1/2^k$, $1/3^k$, $\cdots$, which multiply the second, third, $\cdots$ terms respectively will for large values of k render the sum of those terms negligible compared with the first term if this term is not zero, so that if $C_1 \neq 0$,

$$y_k(s) \sim \left(\frac{a}{\pi}\right)^k C_1 \cos\left(\frac{\pi s}{a} - \alpha_{1k}\right),$$

or if $C_1 = C_2 = \cdots = C_{n-1} = 0$, $C_n \neq 0$, then

$$y_k(s) \sim \left(\frac{a}{n\pi}\right)^k C_n \cos\left(\frac{n\pi s}{a} - \alpha_{nk}\right).$$

A proof may be given by showing that if we write for the latter (general) case

$$y_k(s) = \left(\frac{a}{n\pi}\right)^k C_n \left\{ \cos\left(\frac{n\pi s}{a} - a_{nk}\right) + R_k \right\},$$

then for k sufficiently large R_k is uniformly small on the interval $(-a, a)$. Now we may write

$$R_k = \frac{C_{n+1}}{C_n}\left(\frac{n}{n+1}\right)^k \cos\left(\frac{(n+1)\pi s}{a} - \alpha_{n+1k}\right) + \frac{C_{n+2}}{C_n}\left(\frac{n}{n+2}\right)^k \cos\left(\frac{(n+2)\pi s}{a} - \alpha_{n+2k}\right) + \cdots .$$

Since the Fourier series (1) is convergent, the C's have a maximum absolute value C, and hence throughout the interval $(-a, a)$

$$| R_k | \leqq \frac{Cn^k}{| C_n |} T,$$

where

$$T = \frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \cdots \leqq \int_0^\infty \frac{dx}{(n+x)^k} = \frac{1}{(k-1)n^{k-1}} \cdot$$

It follows that uniformly on the interval $(-a, a)$

$$\lim_{k\to\infty} | R_k | \leqq \lim_{k\to\infty} \frac{Cn}{| C_n | (k-1)} = 0,$$

and our theorem is a consequence.

FRACTIONS*

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Perhaps the author owes an apology to the reader for asking him to lend his attention to so elementary a subject, for the fractions to be discussed in this paper are, for the most part, the halves, quarters, and thirds of arithmetic. But the fact is that the writer has, for some years, been looking on these entities in a somewhat new way. Here will be found a geometric picturization which will be novel to the reader and which will supply a visual representation of arithmetical results of diverse kinds.

The idea of representing a fraction by a circle is one at which the author arrived by an exceedingly circuitous journey. It began with the Group of Picard. In the geometric treatment of this group as carried out by Bianchi in accordance with the general ideas of Poincaré certain invariant families of spheres appear. These spheres, which are found at the complex rational fractions, are mentioned later in this paper. They suggest analogous known invariant families of circles at real rational points in the theory of the Elliptic Modular Group in the complex plane. Finally, it became apparent that this intricate scaffolding of group theory could be dispensed with and the whole subject be built up in a completely elementary fashion. It is this treatment that is undertaken here.

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* Some of the material of this article was presented in an address before the American Mathematical Society at Lawrence, Kansas, November 28, 1936. Other parts have been given in lectures at the Rice Institute, the University of Texas, Northwestern University, and the Armour Institute.

1. The geometric representation. We begin with real fractions. These are usually represented, along with real irrationals, by points on a line. Let this line be the x-axis in an xy-plane of rectangular coördinates.

Through each rational point $x=p/q$, where p and q are integers and the fraction is in its lowest terms, we construct a circle of radius $1/(2q^2)$ tangent to the x-axis and lying in the upper half-plane. It is this circle, which touches the x-axis at the point usually taken to represent the fraction, which will be the geometric picture of the fraction. The integers are represented by circles of radius 1/2; the fractions 1/3, 2/3, 4/3, etc., by circles of radius 1/18; and so on. Every small interval of the x-axis contains points of tangency of infinitely many of these circles.*

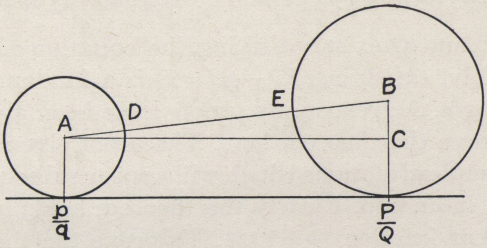

FIG. 1

Let p/q and P/Q be two different fractions in their lowest terms. Consider the distance between the centers of their representative circles. In the figure the horizontal distance AC is $|(P/Q)-(p/q)|$ and the vertical distance CB is the difference of the radii, $|(1/2Q^2)-(1/2q^2)|$. We have

$$AB^2 = \left(\frac{P}{Q}-\frac{p}{q}\right)^2+\left(\frac{1}{2Q^2}-\frac{1}{2q^2}\right)^2$$

$$= \left(\frac{1}{2Q^2}+\frac{1}{2q^2}\right)^2+\frac{(Pq-pQ)^2-1}{Q^2q^2}$$

$$= (AD+EB)^2+\frac{(Pq-pQ)^2-1}{Q^2q^2}.$$

---

* Families of circles tangent to the real axis at the rational points are intimately involved in the geometry of the modular group and in the theory of quadratic forms. In these connections they have been used for some time. Circles defined as in the text except for the more general radius $1/(2hq^2)$ were used, with various values of h, by the present author in papers in the Proceedings of the Edinburgh Mathematical Society in 1917. They were probably used earlier by others.

These circles are called "Speiser circles" by some writers. This name appears to be due to a note of a dozen lines by A. Speiser in the Actes de la Société Helvétique des Sciences Naturelles in 1923. Mention should also be made of a very interesting ninety-page booklet by Züllig, Geometrische Deutung unendlicher Kettenbrüche (1928).

If $|Pq-pQ|>1$, then $AB>AD+EB$, and the two circles are wholly external to one another.

If $|Pq-pQ|=1$, then $AB=AD+EB$, and the two circles are tangent.

If $|Pq-pQ|<1$ we have, since $Pq-pQ$ is an integer, that $Pq-pQ=0$. From this $P/Q=p/q$, which is contrary to the assumption that the fractions are different. So this last inequality is not possible. We may state the following result:

THEOREM 1. *The representative circles of two distinct fractions are either tangent or wholly external to one another.*

2. Adjacent fractions. We shall call two fractions p/q and P/Q *adjacent* if their representative circles are tangent. The condition for this is that $|Pq-pQ|=1$. (We assume here and henceforth that fractions appear in their lowest terms.) We shall prove

THEOREM 2. *Each fraction p/q possesses an adjacent fraction.*

That integers P and Q exist satisfying the equation $|Pq-pQ|=1$ (or, arranging signs suitably, satisfying $Pq-pQ=1$) is a fundamental proposition of the theory of numbers. A great many proofs have been given. For the sake of completeness we prove the theorem here. The proof given is so arranged as to cover the case of complex fractions which will appear subsequently in this paper.

The proof is by induction. Clearly the theorem holds for $|q|=1$, since $p/1$ then has the adjacent fraction $(p+1)/1$. To prove the theorem in general we assume that all fractions whose denominators are less in absolute value than $|q|$ possess adjacent fractions and prove it for the fraction p/q.

Let n be the integer nearest to p/q, whence we can write, with m integral,

$$\frac{p}{q}=n+\frac{m}{q}=\frac{nq+m}{q}, \qquad 0<|m|<|q|.$$

Since $|m|<|q|$, then q/m has an adjacent fraction r/s, so $|sq-rm|=1$. Then the fraction

$$\frac{P}{Q}=n+\frac{s}{r}=\frac{nr+s}{r}$$

is adjacent to p/q; for

$$|Pq-pQ|=|(nr+s)q-(nq+m)r|=|sq-rm|=1.$$

This establishes the theorem.

We can now give a formula for all fractions adjacent to p/q.

THEOREM 3. *If P/Q is adjacent to p/q then all fractions adjacent to p/q are*

$$\frac{P_n}{Q_n}=\frac{P+np}{Q+nq},$$

where n takes on all integral values.

We find readily that the fractions given are adjacent to p/q, for

$$|(P+np)q-p(Q+nq)|=|Pq-pQ|=1.$$

We find that P_n/Q_n and P_{n+1}/Q_{n+1} are adjacent to one another, since

$$|[P+np][Q+(n+1)q]-[P+(n+1)p][Q+nq]|=|Pq-pQ|=1.$$

The circles corresponding to these fractions form a ring around the circle of p/q, all tangent to the circle of p/q, and each tangent to the circles which precede and follow it in the sequence (Figure 2). That this ring completely surrounds the circle of p/q we see from

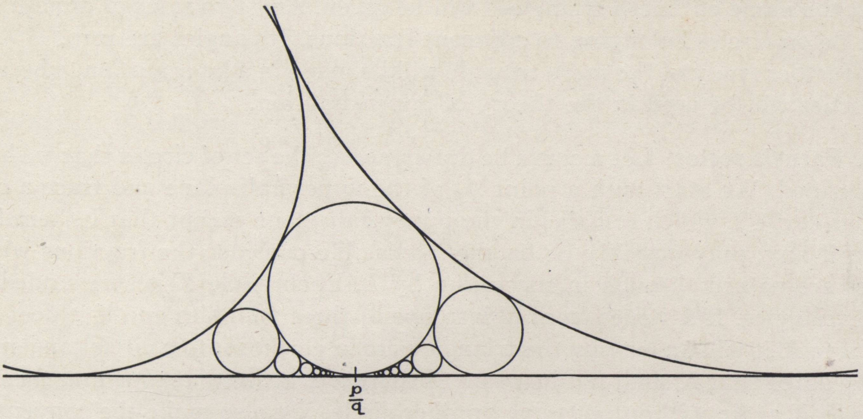

FIG. 2

$$(a) \qquad \frac{P_n}{Q_n}=\frac{p}{q}+\frac{Pq-pQ}{q(Q+nq)}=\frac{p}{q}\pm\frac{1}{q^2(n+Q/q)}\cdot$$

When $n\rightarrow+\infty$, P_n/Q_n approaches p/q from one side; when $n\rightarrow-\infty$, P_n/Q_n approaches p/q from the other side.

It is obvious from the geometric picture that it is not possible to draw a circle lying in the upper half-plane, touching the x-axis, and tangent to the circle of p/q but not intersecting the circles of the ring surrounding the circle of p/q. It follows that there are no further fractions adjacent to p/q.

THEOREM 4. *Of the fractions adjacent to p/q ($|q|>1$) exactly two have denominators numerically smaller than q.*

That two circles of the preceding ring about the circle of p/q are larger than that circle is fairly evident from the geometrical picture. We see also that

$$|Q+nq|<|q|, \qquad \text{or} \qquad |n+Q/q|<1$$

for exactly two values of n; namely, those integers between which $-Q/q$ lies.

For one of these two values of n, $n+Q/q$ is positive, for the other, negative. We see then from (a) that one of the fractions of Theorem 4 is greater than p/q, the other is less.

We shall add one further circle to our collection. The fraction $1/0$ is formally adjacent to the integers $p/1$, since $1\cdot 1-p\cdot 0=1$. We take as its circle the line $y=1$, which touches the circles of all the integers. We consider as the interior of the circle that part of the plane above the line.

3. The mesh triangles. The parts of the upper half plane exterior to all the circles of the system consist of an infinite number of circular arc triangles to which the name of "mesh triangles" will be given. Any two sides of a mesh triangle lie on circles belonging to adjacent fractions. Its angles are zero.

A further study of the mesh triangle will be made in a later section, where its properties will be used in the theory of approximation.

4. Farey's series. Let a curve be drawn across the set of circles that we have just defined. We start with a point A_0 of the upper half-plane and trace a continuous curve L which remains in the upper half plane except that its terminal point, if any, may possibly lie on the x-axis. We consider the fractions whose circles are passed through in succession by L. Each circle K is surrounded by mesh triangles. If L issues from K into one of these triangles and if it does not return to K it will on leaving the triangle pass in general into a circle tangent to K. We agree that K shall not be counted twice if L passes out of K and then returns to K again without entering another circle. We also make the convention that if L touches two circles at their point of tangency without entering either then one or the other shall be considered as crossed by L (the larger circle, say, or the one on the left, or the one whose fraction is the greater). We can then state as an established proposition the following:

PRINCIPLE. *If two circles of the system are penetrated in succsssion by L then the corresponding two fractions are adjacent.*

As a first illustration let L be a line, $y=k$, parallel to the x-axis, starting say at a point on the positive y-axis and stopping at $x=1$. The points of tangency with the x-axis of the circles through which L passes are arranged in order from left to right; that is, the corresponding fractions are arranged in numerical order. If $1/(n+1)^2<k<1/n^2$, L intersects the circles of all fractions in the interval $0\leqq x\leqq 1$ whose denominators do not exceed n and the circles of no other fractions. These fractions arranged in numerical order constitute what is known as a Farey's series of order n for the interval.

Certain theorems concerning Farey's series are now readily established.* Let $p/q<p'/q'$ be successive fractions of the series, all the numbers being positive.

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* See, for example, Landau, Vorlesungen über Zahlentheorie, 1927, Band I, pp. 98–100. Farey discussed these fractions more than a century ago.

(1) $p'q-pq'=1$. This results from the Principle stated earlier in this section.

(2) $q+q'\geqq n+1$. For $(p+p')/(q+q')$ is a fraction between p/q and p'/q'; since it does not belong to the series its denominator is greater than n.

(3) For any number ω of the interval there is a fraction p/q of Farey's series of order n such that

$$\left|\frac{p}{q}-\omega\right| \leqq \frac{1}{(n+1)q}.$$

The number ω lies in an interval formed by a Farey's fraction p/q and an adjacent fraction $(p+p_1)/(q+q_1)$ not of the series (p_1/q_1 being the Farey's fraction preceding or following p/q). The length of this interval is $1/(q+q_1)q$, where $q+q_1\geqq n+1$.

If we take other simple forms for the curve L we get series analogous to Farey's. Thus if we take a line with a positive slope starting above $y=1$ and terminating at the origin we have all non-negative fractions such that

$$pq \leqq n,$$

where n is suitably chosen. If these be arranged in order of numerical magnitude then successive fractions satisfy (1) above; (2) is replaced by $(p+p')(q+q')\geqq n+1$.

The series such that $0<p\leqq n$ may be got by taking for L the arc of a suitable circle tangent to the x-axis at the origin.

5. The problem of approximation. Dirichlet showed by elementary means that if ω is a real irrational number then the inequality

$$\left|\frac{p}{q}-\omega\right| < \frac{k}{q^2} \tag{1}$$

is satisfied by infinitely many fractions p/q if $k=1$. If, however, $\omega=r/s$, a rational, then the inequality is satisfied by only a finite number of rationals p/q however large k be chosen. For

$$\left|\frac{r}{s}-\frac{p}{q}\right| = \left|\frac{rq-ps}{sq}\right| \geqq \frac{1}{|sq|} = \frac{\left|\frac{q}{s}\right|}{q^2} > \frac{k}{q^2}$$

except for a finite number of values of q for which $|q|\leqq|ks|$; also for each q there is clearly only a finite number of fractions satisfying the inequality.

Various suites of fractions approximating to ω satisfy (1) with suitable k. The convergents in the ordinary continued fraction for ω satisfy the inequality with $k=1$. A suite of Hermite admits the smaller value $k=1/\sqrt{3}$.

The determination of the best value of k is due to Hurwitz,* who proved the following theorem.

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* Mathematische Annalen, vol. 39, 1891, pp. 279–284. See also Borel, Journal de Mathématiques, 5th ser., vol. 9, 1903, pp. 329 ff; L. R. Ford, Proceedings of the Edinburgh Mathematical Society, vol. 35, 1917.

THEOREM 5. *If* $k=1/\sqrt{5}$, *then for each irrational* ω *there are infinitely many fractions* p/q *satisfying the inequality* (1).

If $k<1/\sqrt{5}$, *then there are irrationals* ω *such that only a finite number of fractions* p/q *satisfy* (1).

Let the curve L of the previous section be a vertical line which terminates at the irrational ω. Then L intersects infinitely many circles of the system we are considering and is tangent to none. If L cuts the circle of p/q then the distance from ω to p/q is less than the radius:

$$\left|\frac{p}{q}-\omega\right|<\frac{1}{2q^2}\cdot$$

The curve L thus provides us with an infinite suite of all those fractions satisfying (1) when $k=1/2$. If p/q and p'/q' are successive fractions of this suite then, from the Principle of the preceding section, we have $p'q-pq'=\pm 1$. Also, we see geometrically that $|q'|>|q|$. The convergence of the suite to ω is then immediately proved.

We propose now to prove Hurwitz' theorem by a study of the circles of our system. This elementary proof will be free of continued fractions on the one hand and of the theory of the Modular Group on the other.

The line L cuts across infinitely many mesh triangles. The first part of the theorem of Hurwitz is a consequence of the following result which will now be proved:

THEOREM 6. *Of the three fractions whose circles form the boundary of a mesh triangle which* L *crosses, at least one satisfies* (1) *with* $k=1/\sqrt{5}$.

We consider the mesh triangle in some detail. Let P/Q, p/q, p_1/q_1 be the fractions whose circles bound the area, where

$$0<Q\leqq q<q_1=q+Q,\quad p_1=p+P.$$

We suppose, to fix the picture, that the largest of the three circles is on the right; that is, $P/Q>p/q$, whence $Pq-pQ=+1$. If not, we could make suitable changes of sign in the following; or, more simply, we could reflect the entire figure in the y-axis by changing the signs of the numerators and of ω and at the conclusion of our analysis we could reflect again.

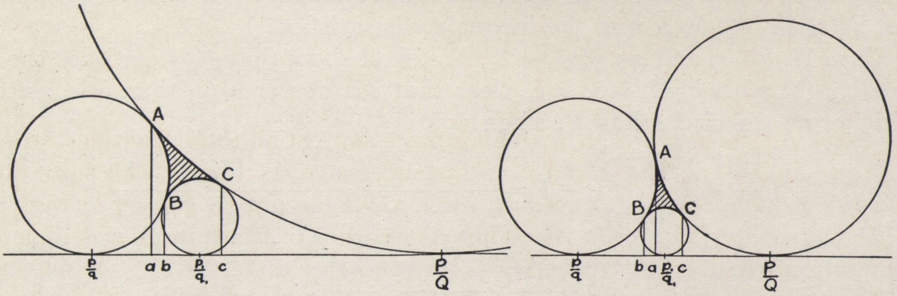
The vertex A (see Figure 3) is the point of contact of the circles of p/q and P/Q. It divides the line of centers of these circles in the ratio of their radii, $1/2q^2:1/2Q^2$, or $Q^2:q^2$. The abscissa of this point is found by an early formula of analytic geometry, or by the methods of high school plane geometry, to be

$$a=\frac{q^2\cdot\dfrac{p}{q}+Q^2\cdot\dfrac{P}{Q}}{q^2+Q^2}=\frac{pq+PQ}{q^2+Q^2}\cdot$$

The abscissas of B and C, the other vertices, may be written down by an interchange of letters,

$$b = \frac{pq + p_1q_1}{q^2 + q_1^2}, \qquad c = \frac{PQ + p_1q_1}{Q^2 + q_1^2}.$$

In order that L cross the mesh triangle under consideration it is necessary and sufficient that ω lie in the interval whose right end is c and whose left end



Case I *Case* II

FIG. 3

is the lesser of the two quantities a and b. We find

$$a - \frac{p}{q} = \frac{Q}{q(q^2 + Q^2)}, \qquad b - \frac{p}{q} = \frac{q_1}{q(q^2 + q_1^2)},$$

whence subtracting and remembering that $q_1 = q + Q$, we have

$$b - a = \frac{q^2 - qQ - Q^2}{(q^2 + Q^2)(q^2 + q_1^2)}.$$

We put $q/Q = s \geqq 1$. The sign of $b - a$ is (dividing the preceding numerator by Q^2) the same as the sign of $s^2 - s - 1$. Now, factoring,

$$s^2 - s - 1 = \left(s + \frac{\sqrt{5} - 1}{2}\right)\left(s - \frac{\sqrt{5} + 1}{2}\right),$$

and the square root of 5 first enters the picture. Here the first factor is positive and the sign is determined by the second factor. We have two cases to consider.

Case I. $a < b$, and * $s > \frac{1}{2}(\sqrt{5} + 1)$. We shall show that P/Q satisfies the inequality (1) with $k = 1/\sqrt{5}$. We have

$$\left|\frac{P}{Q} - \omega\right| < \frac{P}{Q} - a = \frac{q}{Q(q^2 + Q^2)} = \frac{s}{s^2 + 1} \cdot \frac{1}{Q^2}.$$

---

* Here and in many subsequent relations the sign of equality is not possible, for one member is rational and the other is irrational.

If now

$$\frac{s}{s^2+1} > \frac{1}{\sqrt{5}}$$

we have

$$s^2 - \sqrt{5}\,s + 1 < 0, \qquad \left(s - \frac{\sqrt{5}+1}{2}\right)\left(s - \frac{\sqrt{5}-1}{2}\right) < 0,$$

which is impossible since both factors in the first member are positive. It follows that $s/(s^2+1) < 1/\sqrt{5}$, and the inequality is satisfied.

Case II. $b < a$, and $s < \frac{1}{2}(\sqrt{5}+1)$. We shall show that p_1/q_1 satisfies the inequality (1) with $k = 1/\sqrt{5}$. It is clear that c is nearer p_1/q_1 than b is, since C is higher on the circle than B. Then

$$\left| \frac{p_1}{q_1} - \omega \right| < \frac{p_1}{q_1} - b = \frac{q}{q_1(q^2+q_1^2)} = \frac{s(s+1)}{s^2+(s+1)^2} \cdot \frac{1}{q_1^2} \cdot$$

If

$$\frac{s(s+1)}{s^2+(s+1)^2} > \frac{1}{\sqrt{5}},$$

then

$$(\sqrt{5}-2)s^2 + (\sqrt{5}-2)s - 1 > 0,$$

or, dividing,

$$s^2 + s - \sqrt{5} - 2 > 0.$$

Factoring,

$$\left(s - \frac{\sqrt{5}+1}{2}\right)\left(s + \frac{\sqrt{5}+3}{2}\right) > 0,$$

which is impossible since the first factor is negative and the second is positive.

We have proved that one, at least, of the three fractions whose circles bound the mesh triangle satisfies (1) with $k = 1/\sqrt{5}$. Since L passes through infinitely many mesh triangles it follows that infinitely many rational fractions satisfy the inequality with $k = 1/\sqrt{5}$.

It remains to exhibit an irrational ω for which (1) holds for only a finite number of fractions for $k < 1/\sqrt{5}$. We take, in fact, $\omega = \frac{1}{2}(\sqrt{5}+1)$ and any $h < 1$ and show that there is a finite number of fractions for which

$$\left| \frac{p}{q} - \frac{\sqrt{5}+1}{2} \right| < \frac{h}{\sqrt{5}q^2} \cdot \tag{2}$$

For a fraction satisfying this we may write

$$\frac{p}{q} - \frac{\sqrt{5}+1}{2} = \frac{\theta}{\sqrt{5}q^2},$$

where $|\theta| < h < 1$. Writing this

$$\frac{p}{q} - \frac{1}{2} = \frac{\sqrt{5}}{2} + \frac{\theta}{\sqrt{5}q^2},$$

squaring, and rearranging, we get

$$5q^2[(p^2 - pq - q^2) - \theta] = \theta^2.$$

The expression in square brackets must be positive, whence the integer $p^2 - pq - q^2$, which cannot be zero (since then p/q would turn out irrational) must equal or exceed 1. We have

$$q^2 = \frac{\theta^2}{5[(p^2 - pq - q^2) - \theta]} < \frac{h^2}{5(1 - h)}.$$

This limits q to a finite number of values. For each q, the inequality (2) then limits p; and the number of fractions is finite.

Other irrationals might have been used here; for example, $\omega = (r\sqrt{5} + s)/t$, where r, s, t are integers. These last irrational numbers are found in every interval of the x-axis.

6. Continued fractions. One of the most successful interpretations supplied by our system of circles is the geometric picture of the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}; \tag{3}$$

or, as often written,

$$a_0 + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\,\frac{1}{a_3 +} \cdots$$

Here the quantities a_n are integers. If all the integers are positive, except possibly a_0, the continued fraction is *simple.*[*]

The nth *convergent*, p_n/q_n, is the quantity that remains when all that part of the expression following a_{n-1} is erased. We have

$$\frac{p_1}{q_1} = \frac{a_0}{1}, \qquad \frac{p_2}{q_2} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1},$$

$$\frac{p_3}{q_3} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{(a_0 a_1 + 1)a_2 + a_0}{a_1 a_2 + 1} = \frac{a_2 p_2 + p_1}{a_2 q_2 + q_1}.$$

---

[*] So called by Chrystal in whose Algebra, Part II, pp. 396ff, will be found one of the best elementary treatments of continued fractions in English.

Here and in the following we are using the p's and q's to represent the actual numerators and denominators of the fractions in the second members and not merely numbers proportional thereto.

The last equation exhibits for $n=2$ the recurrence formula from which we calculate the convergents step by step,

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_n p_n + p_{n-1}}{a_n q_n + q_{n-1}}. \tag{4}$$

This formula is readily established by induction. It holds, we have just seen, for $n=2$; assuming it up to any later n we prove it for $n+1$. We get p_{n+2}/q_{n+2} from p_{n+1}/q_{n+1} by replacing a_n by a_n+1/a_{n+1};

$$\frac{p_{n+2}}{q_{n+2}} = \frac{(a_n + 1/a_{n+1})p_n + p_{n-1}}{(a_n + 1/a_{n+1})q_n + q_{n-1}} = \frac{a_{n+1}(a_n p_n + p_{n-1}) + p_n}{a_{n+1}(a_n q_n + q_{n-1}) + q_n} = \frac{a_{n+1}p_{n+1} + p_n}{a_{n+1}q_{n+1} + q_n}.$$

This is the required formula with n replaced by $n+1$ and (4) holds for $n \geqq 2$. It holds also for $n=1$ if we take $p_0/q_0=1/0$.

If the continued fraction terminates, its value is the last convergent. If there are infinitely many a's the value to be assigned to the fraction is lim p_n/q_n if this limit exists.

We investigate the suite of convergents

$$\frac{p_0}{q_0}\left(=\frac{1}{0}\right),\quad \frac{p_1}{q_1}\left(=\frac{a_0}{1}\right),\quad \frac{p_2}{q_2},\quad \frac{p_3}{q_3},\ \cdots .$$

THEOREM 7. *Successive convergents,* p_n/q_n *and* p_{n+1}/q_{n+1}, *are adjacent fractions.*

From (4)

$$p_{n+1}q_n - q_{n+1}p_n = (a_n p_n + p_{n-1})q_n - (a_n q_n + q_{n-1})p_n = -(p_n q_{n-1} - q_n p_{n-1}).$$

Since $p_1 q_0 - p_0 q_1 = a_0 \cdot 0 - 1 \cdot 1 = -1$ we have that

$$p_{n+1}q_n - q_{n+1}p_n = (-1)^{n+1},$$

which establishes the theorem.

Given the tangent circles of p_{n-1}/q_{n-1} and p_n/q_n and the integer a_n, how do we find the circle of p_{n+1}/q_{n+1}? For all integral a_n we get from (4) (see Theorem 3, Section 2) the ring of tangent circles around the circle of p_n/q_n. For $a_n=0$ we get the circle of p_{n-1}/q_{n-1}; for $a_n=1$ we get a circle next to this; for $a_n=2$ we get the next around the ring; and so on. For $a_n=-1, -2,$ *etc.* we pass around in the opposite direction.

To determine the direction consider

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^{n+1}}{q_{n+1}q_n} = \frac{(-1)^{n+1}}{(a_n q_n + q_{n-1})q_n} = \frac{(-1)^{n+1}}{q_n^2\left(a_n + \dfrac{q_{n-1}}{q_n}\right)}.$$

For a_n large and positive the last denominator is positive. If $n+1$ is even then p_{n+1}/q_{n+1} is to the right of p_n/q_n and we have counted around the circle of p_n/q_n in a clockwise direction; if $n+1$ is odd p_{n+1}/q_{n+1} is less than p_n/q_n and we have counted in a counter-clockwise direction.

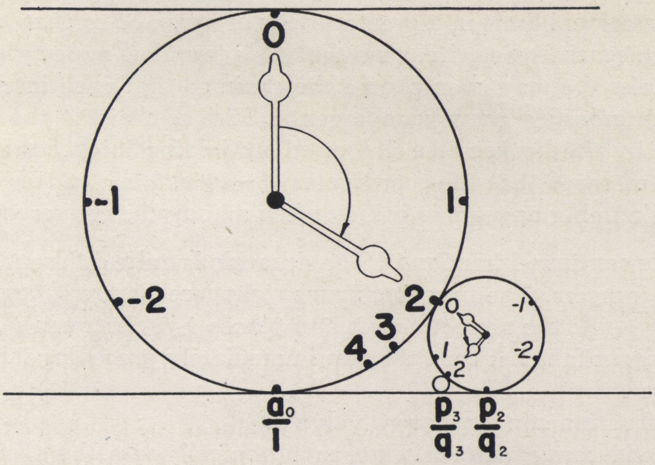


FIG. 4

We may liken the circle of p_n/q_n to the face of a clock (Figure 4). Set the hand to point toward the center of the circle of p_{n-1}/q_{n-1}. It is now in the zero position. If we seek an even convergent turn clockwise to the a_nth point of tangency; if we seek an odd convergent turn counter-clockwise to the a_nth point of tangency. If a_n is negative the turning is in the opposite direction in each case. The hand now points to the center of p_{n+1}/q_{n+1}.

Initially $(n=1)$ we set the hand to point vertically upward to the point of tangency of $y=1$ with the circle of $a_0/1$, as in the figure. The positive direction is clockwise. On the next clock the positive direction is counter-clockwise; on the next, clockwise; and so on. In the figure $a_1=2$, $a_2=2$, and if $a_0=0$ the first four convergents, starting with p_0/q_0, are

$$\frac{1}{0}, \quad \frac{0}{1}, \quad \frac{1}{2}, \quad \frac{2}{5}.$$

We see then that *the circles of the convergents, in order, of a continued fraction form a sequence, or chain, of circles. The chain begins with the circle $y=1$ and each circle is tangent to the circles preceding and following it in the sequence.*

Conversely, any such chain has a unique corresponding continued fraction. We determine a_0 as the integer whose circle touches the first circle, $y=1$, in the chain; and a_2, a_3, etc., are got uniquely by counting points of tangency, as already explained.

A chain can be defined in many ways and with much arbitrariness. A simple way is to use a curve L, starting above $y=1$, as explained in Section 4. We shall not object to $a_n=0$, which gives $p_{n+1}/q_{n+1}=p_{n-1}/q_{n-1}$, nor to $q_n=0$, which means that $y=1$ reappears as a circle of the chain.

The preceding picture answers various questions about convergence. We see that it is possible so to choose the chain that p_n/q_n approaches a prescribed rational or irrational limit in a great variety of ways, or so that p_n/q_n becomes positively or negatively infinite, or so that p_n/q_n wanders more or less arbitrarily over the x-axis. We may even choose the chain so that each interval of the x-axis contains infinitely many convergents. Such a chain is the set of circles passed over by the broken line L formed by joining in succession the points $A_1, A_2, \cdots$ where A_n has the coördinates $[(-1)^n n, 1/n]$. In this example each real rational number appears infinitely often among the convergents.

7. Simple continued fractions.* The situation is quite different if all $a_n (n>0)$ are positive integers. The first clock runs right (see Figure 4) and makes p_2/q_2 greater than p_1/q_1; the next runs left and places p_3/q_3 between p_1/q_1 and p_2/q_2. Each convergent falls between the two immediately preceding it. Certain facts then appear:

(*a*) The fraction converges to a value ω.

(*b*) The odd convergents increase monotonically to ω, the even convergents decrease monotonically to ω.

(*c*) ω lies between each pair of successive convergents.

(*d*) $q_{n+1} > q_n \qquad (n > 1).$

(*e*) $$\left| \frac{p_n}{q_n} - \omega \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{a_n q_n^2} \leqq \frac{1}{q_n^2}.$$

Here the second expression is the distance between the convergents p_n/q_n and p_{n+1}/q_{n+1}, and $q_{n+1}=a_n q_n+q_{n-1}>a_n q_n$ $(n>1)$.

(*f*) $$\left| \frac{p_n}{q_n} - \omega \right| > \frac{a_{n+1}}{q_n q_{n+2}},$$

for the second member is the distance between p_n/q_n and p_{n+2}/q_{n+2}.

We see geometrically how to find the simple continued fraction for a given ω. We select for a_0 the integer next below ω (see Figure 4). We then turn the clock to the right until p_2/q_2 lies as far to the left as possible without passing ω, thus finding a_1. We turn the second clock to the left until p_3/q_3 is as far to the right as possible without passing ω, whence we have a_2; and so on.

That the development is unique is apparent. The first a_n's that differ in two developments throw the values of the two continued fractions into different intervals thereafter. There is an unimportant exception in the case of a fraction which terminates. A last $a_n>1$ may be written $(a_n-1)+1/1$, thus inserting an additional convergent.

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* Züllig, *loc. cit.*, has the chain of circles given here with figures for many numerical cases.

It is not difficult to identify this process of forming the continued fraction for ω with the usual arithmetic process: (1) taking a_0 as the largest integer in ω; (2) forming ω_1, the reciprocal of the remainder $\omega-a_0$, and taking a_1 as the largest integer in ω_1; (3) taking a_2 as the largest integer in ω_2, the reciprocal of ω_1-a_1; and so on.

We see geometrically that p_n/q_n is a good approximation to ω if a_n is very large, for our clock has then turned far toward the bottom. This is clear, of course, from (*e*) above. Thus in the continued fraction

$$\pi = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots,$$

the second convergent 22/7 and the fourth 355/113 (here $a_4=292$) are especially good and are, in fact, much used.

Suppose we take $a_n=1$ for all n in order to avoid a good approximation as far as we can. We have

$$\omega = 1 + \frac{1}{1+}\,\frac{1}{1+}\cdots = 1 + \frac{1}{\omega},$$

or $\omega^2-\omega-1=0$. From this, since $\omega>1$,

$$\omega = \tfrac{1}{2}(\sqrt{5}+1).$$

This is the irrational quantity we used in the latter part of the proof of Hurwitz's theorem.

8. Complex fractions.* Complex numbers $x+iy$, where x and y are real and $i=\sqrt{-1}$, are commonly represented by points (x, y) in an xy-plane using rectangular coördinates. The complex integers, $n=n'+in''$, where n' and n'' are real integers, form a square lattice in the plane. A fraction is the quotient of two integers

$$\frac{p}{q} = \frac{p'+ip''}{q'+iq''} = \frac{p'+ip''}{q'+iq''}\cdot\frac{q'-iq''}{q'-iq''} = \frac{p'q'+p''q''}{q'^2+q''^2} + i\,\frac{p''q'-p'q''}{q'^2+q''^2}$$

and is represented by the point with the coördinates shown in the last member. The complex fractions thus consist of the numbers $x+iy$, where x and y are real fractions. We think of the xy-plane as being horizontal, and we introduce a third or z-axis perpendicular to the x- and y-axes.

The analogue of the circle of Section 1 is a sphere. Through the point in the plane which represents the fraction p/q (in its lowest terms) we construct a sphere touching the complex plane there, lying in the upper half-space, and having the radius $1/(2q\bar{q})$. Here $\bar{q}$ is the conjugate of q, $\bar{q}=q'-iq''$; whence

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* For the matters treated in this section see L. R. Ford, Transactions of the American Mathematical Society, vol. 19, 1918, pp. 1–42.

$q\bar{q}=q'^2+q''^2=|q|^2$. This sphere will be the geometric representation of p/q. Every small area in the complex plane contains points of tangency of infinitely many of these spheres.

Interpreting Figure 1 as a pair of spheres at complex fractions, we have

$$AB^2 = \left|\frac{P}{Q}-\frac{p}{q}\right|^2+\left(\frac{1}{2Q\bar{Q}}-\frac{1}{2q\bar{q}}\right)^2=\left(\frac{1}{2Q\bar{Q}}+\frac{1}{2q\bar{q}}\right)+\frac{|Pq-pQ|^2-1}{Q\bar{Q}q\bar{q}}.$$

If $|Pq-pQ|>1$, then $AB>AD+EB$, and the spheres are wholly external to one another. If $|Pq-pQ|=1$ the spheres are tangent. In no case do the spheres intersect. We call the fractions *adjacent* if the spheres are tangent.

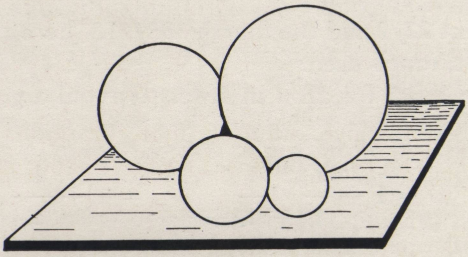

FIG. 5

That p/q has an adjacent fraction P/Q we already know, for we have designed the proof of Theorem 2 to cover this case. Then the fraction

$$\frac{P_n}{Q_n}=\frac{P+np}{Q+nq}$$

of Theorem 3, where n is any complex integer, is adjacent to p/q. We note that the four fractions P_m/Q_m, where $m=n+1$, $n-1$, $n+i$, $n-i$, are adjacent to P_n/Q_n; for $|(P+np)(Q+mq)-(P+mp)(Q+nq)|=|m-n|\cdot|Pq-pQ|=1$. That is, each sphere tangent to a given sphere touches four other spheres tangent to the given sphere. It is not difficult to show that the fractions P_n/Q_n constitute the complete set of fractions adjacent to p/q.

How many of the spheres touching the sphere of p/q are larger than that sphere? We require the number of solutions of

$$|Q+nq|<|q|, \qquad \text{or} \qquad |n+Q/q|<1.$$

Now $|n+Q/q|$ is the distance from the integer n to the point $-Q/q$. The problem takes the form: If a unit circle is drawn in the complex plane, how many complex integers does it contain? For various positions of the center $(-Q/q)$ the answer is 2, 3, or 4, as is readily seen. We put aside the case $|q|=1$, which puts the center at an integer, when clearly there are no larger tangent spheres. We introduce the plane $z=1$ to touch the spheres of the integers and be the sphere of their adjacent fraction 1/0.

The use of complex integers in the continued fraction (3) offers no difficulty. We are led to the geometric picture of the continued fraction as a chain of spheres, beginning with the plane $z=1$, and proceeding thence from sphere to tangent sphere. We have here the same possibilities of convergence or divergence as before. At the worst we can set up continued fractions whose convergents are present in every small region of the complex plane. There is no immediate generalization of the simple continued fraction to the complex case. There are, however, schemes for developing a complex number in a continued fraction, the best processes being due to Hurwitz.*

The analogue of Hurwitz's theorem on rational approximations to an irrational number was first discovered by the present author.† The inequality (1) is replaced by

$$\left| \frac{p}{q} - \omega \right| < \frac{k}{q\bar{q}},$$

and the minimum k for which infinitely many fractions always satisfy the inequality is $k=1/\sqrt{3}$. The proof of this, however, is not elementary.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This Department of the MONTHLY *has been created as an experiment to afford a place for the discussion of the place of mathematics in education. With this topic will naturally be associated other matters emphasizing the educational interests of those who teach mathematics. It is not intended to take up minute details of teaching technique. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. The success of this department obviously will depend upon the cooperation of the readers of the* MONTHLY. *Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

MATHEMATICAL EDUCATION IN GERMANY BEFORE 1933

RICHARD COURANT, New York University

At the suggestion of the editors I shall try to give a brief account of trends in the teaching of mathematics in Germany, particularly at German universities, in the period from the World War until 1933.

The situation of mathematics in Germany, since the early part of the 19th century, has been pivoted around a definite connection between university and high school. In a sweeping reform following the French Revolution, the institution of the German "humanistisches Gymnasium" was established. Teachers at these institutions were required to undergo a very thorough academic preparation, and the task of preparing them was entrusted to the philosophical faculties of the universities. The teachers' training was in no way of an elementary character. First rate mathematical scholars—following the example of classical philologists—gathered a crowd of enthusiastic students and lectured on the most advanced subjects on the boundary line of research. The pace was set by Jacobi, who discussed in his class, ten hours a week, his newest discoveries on elliptic functions.

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* Acta Mathematica, vol. 11, 1887, pp. 187–200.

† L. R. Ford, Transactions of the American Mathematical Society, vol. 27, 1925, pp. 146–154.

The requirements of the state board examination for high school teachers were excessively high; they called for full-fledged scholars and probably could never be enforced even approximately. Still, published yearly reports of the better high schools sometimes contained important mathematical contributions from the staff, as for instance the famous paper on Abelian functions by Weierstrass. A considerable number of the famous German mathematicians of the 19th century started their careers as more or less successful teachers in high schools.

For the development of mathematics at universities it certainly was fortunate that a rather stationary and responsive audience was assured. But the young teacher, coming from the university to the school, which he had left five or six years ago, found little use for his specialized knowledge of Abelian functions or other sophisticated subjects, while he had more or less forgotten the material to be covered in the high school curriculum.

However, on the whole, the system worked quite satisfactorily as long as the number of teachers and of high school students was comparatively small, and the latter somewhat restricted to a group seriously striving for a higher scholastic education. But gradually this situation changed with the progress of democratization in education. New types of high schools sprang up, drawing away from the rigorous humanistic ideal and tending more to useful lines. After the war of 1870 this process led to an enormous increase in the demand for teachers, not all of whom were able to attain high scholastic standards. But nothing was changed in the fundamental system of four to six years of graduate study as preparation for school teachers. The number of mathematics chairs at universities increased considerably; even more the number of students in the courses. The contradictory character of the relationship between university and high school became a burning question. What could be done to preserve the university standards and still meet the real needs of a broadening school system?

Under Felix Klein's inspiring leadership progress was made in bridging the gap from both sides. The curriculum of the high schools was modernized. The concepts of function and of limit, the elements of calculus, analytic geometry, projective geometry, *etc.*, conquered much space in the high school curriculum, superseding antiquated routine material. On the other hand, in the university, survey courses were introduced, Felix Klein's lectures on elementary mathematics from a higher viewpoint being the outstanding example. It was very fortunate that also other academic teachers of highest rank, *e.g.*, Hilbert, followed this example enthusiastically. Hilbert's courses in foundations of geometry and in intuitive geometry, and his lectures on the problem of squaring the circle are famous instances.

The process of mutual adaptation was interrupted by the World War, the aftermath of which created intensified problems. The number of high school students leaped to unknown heights. A desperate demand for large numbers of teachers was voiced officially. Thus the classrooms of the universities became overcrowded with students who were not driven by thirst for higher learning but attracted by the prospect of a secure job. Soon the teaching jobs were filled, but the inert stream of students in the universities preparing for the teaching profession continued.

At the University of Berlin in some years more than 600 students registered for a single course in number theory, and about as many in calculus and analytic geometry. At Göttingen courses on the theory of functions of a complex variable had an attendance of up to 330 students. In highly specialized and advanced courses 50–100 students could often be found, many of whom were also preparing for a teaching job. While at the high schools the reform work proceeded organically, the universities were facing a more complex problem. How could they give the masses a really worth-while instruction without hopelessly lowering the general level and without giving up the ideals of academic freedom?

The complexity of the situation was enhanced by different factors: While the great majority of mathematics students still was preparing for the state board examination for teachers (incidentally at the same time studying two minor subjects, usually physics and another science), an increasing number of students was considering other professions, *e.g.*, the actuarial career, and at the principal mathematical centers there was always a group of deeply interested young scholars, partly from abroad, with outspoken academic ambitions. This smaller elite group was apt to form a separate stratum in closer contact with the staff and rather separated from the bulk of the student body. Naturally the professors were more interested in this top group than in the crowd.

Another difficulty arose from the traditional academic freedom in learning. There were no official examinations and no marks all through the years between high school and the state board examination (except examinations for tuition scholarships). It happened that candidates, after five years of attending advanced classes and having had little contact with instructors and fellow students, came up for examination, and to the dismay of both sides it turned out that they knew literally nothing about mathematics and had stupidly wasted their years at the university. The absolute freedom for the student to select his courses and to conduct his studies to his own taste was sometimes a handicap for the majority, although of the highest benefit to the top group. A similar effect came from the academic freedom in teaching. The professor was absolutely free in the selection and presentation of his material, not bound to any curriculum. Thus it actually happened that elementary calculus was taught in a class of several hundred students on the basis of Brouwer's intuitionism, a very interesting experiment for the lecturer and three or four students but less useful for the others.

During the period after 1919 the technological institutes ("Technische Hochschulen") began to compete with the universities in the task of training high school teachers in mathematics and physics. The mathematics classes at these institutions were terribly overcrowded by the enormous number of prospective engineers. But the professors of mathematics were anxious to have also some students interested in mathematics as such, and the technological institutes succeeded in being opened also for teacher training. However, this competition of the technological institutes did not amount to much.

Since there was little uniformity in the different universities, I shall, in some detail, describe only the situation which had developed at Göttingen.

The beginner, entering the university and facing a confusing variety of possibilities, found some advice in a printed guide and, if he desired, could consult with assistants assigned to advisory work. He was not bound to heed suggestions, but usually in the first year he took differential and integral calculus, 4 hours a week, analytic geometry, also 4 hours, and the *Anfängerpraktikum*, practical exercises for beginners, 3–5 consecutive hours a week. (In addition a 5 hour lecture course on experimental physics, and other courses.) The courses never followed closely a definite textbook. The students were supposed to take and work out notes or to study the official record of the lectures in the reading room. The level of the calculus courses corresponded more or less to an advanced calculus course at American universities.

It was a strict rule in the science faculty that these beginners' courses be given by experienced, responsible professors and not by young instructors. But a course was hardly ever repeated by the same professor in consecutive years. Thus entirely different, even antagonistic tendencies in presentation were offered to the student, who was urged to supplement his information by independent study of the literature easily accessible in the large mathematical reference library. While the lecture courses followed more or less the old tradition of separation between the talking professor and the passive audience, the five hour period of practical exercises was an important innovation for coping with the problem of mass instruction. The students, more than 200, were given in advance a mimeographed sheet with problems, some mere exercises, some requiring a certain amount of inventive thinking. They were supposed to come to class after having already given some thought to the problems, while the professor in charge, prior to the class meeting, held a conference with a group of assistants (about one for every 15–20 students) concerning the problems, the different methods and aspects of the solutions, and the probable difficulties of the students. The class room consisted of two large halls with big tables and blackboards, and the 5 hours (or less if the student preferred) were spent not only in finding the solutions, but mainly in personal discussion between students and assistant instructors. There was ample opportunity for anyone who cared to clarify any subject connected with the beginners' courses in discussion with competent people. The solutions were later written up by the students, handed in, criticized in writing by a conference of the assistants, and returned and individually discussed at the next meeting. Records with marks were taken. This institution proved astonishingly efficient. A considerable percentage of the students really learned something; others realized that it was best for them to give up mathematics, and soon the few with talents were discovered and given particular consideration. Forming of study groups among the students was encouraged, and collusion in solving the problems was not discouraged.

After the beginners' courses, which were sometimes supplemented by a course on descriptive geometry, the period of the normal full courses, "Kursusvorlesungen," started. A great variety was offered here: Theory of functions of a complex variable, differential equations, functions of real variables, potential theory, partial differential equations of physics, analytic dynamics, algebra, elementary number theory, differential geometry, projective and non-euclidean geometry, calculus of variations, group theory, mechanics of continuous bodies, and many other subjects were covered. The average student took two or three courses a term, altogether often more than 12 of these before his examination. Theory of functions of a complex variable was always included, but generally there was much leeway left in the selection.

In the same category belonged the full courses on elementary mathematics from a higher viewpoint, and other survey courses. Many of these "Kursusvorlesungen" again were attended by hundreds of students, and again an attempt was made to reach at least a part of them personally in additional two hour periods of exercises ("Übungen"), often held by one or more assistants. Here problems of a more advanced type were discussed, sometimes in writing, but not on such a systematic basis as in the beginners' "Praktikum."

At Göttingen this method of supplementing lecture courses by "Übungen" for larger groups of students was made possible by the institution of the mathematical assistant. Before the war, assistantships in mathematics hardly existed except for a few such positions at Göttingen and Berlin. Future professors were recruited from the ranks of the "Privatdozenten," voluntary academic teachers admitted to teaching only after a very careful process of selection, but not paid except by student fees. They came mostly from the moderately well-to-do middle class and were more or less able to support themselves. Only at some larger institutions, like Göttingen and Berlin, did income from student fees suffice to support a few of the less wealthy "Privatdozenten," and, in certain exceptional cases, government fellowships, *etc.*, made academic careers possible for promising students without means. After the destruction of middle class wealth by war and inflation, it became imperative to put the financial care for the next academic generation on a more systematic basis. Thus most "Privatdozenten" were paid some modest salary by making them assistants or giving them some official duties in the teaching process, and it was only by this definite innovation that the problem of instruction for larger masses could be tackled. At the university of Göttingen, for example, where, before the war, only one full-time and one or two part-time assistants in mathematics existed, there were in 1929 more than 10 full-time assistants, all actual or prospective "Privatdozenten," and in addition an indefinite number of part-time assistants according to the needs of the "Praktikum" and "Übungen." Generally also these part-time assistants were fully trained young scholars with the doctor's degree and some record of independent research.

After a student had taken a few of the "Kursusvorlesungen," he usually started attending some courses of the highest category, so-called special lectures and seminars. These were mostly two hour courses, and covered an enormous variety of subjects according to the personal taste and interest of the professor or instructor. Very often these special courses concerned the particular field in which the instructor was doing research and led directly to or beyond the boundary line of general knowledge. It was mostly in connection with these special courses and seminars that actual research was done by teachers and students. Little special consideration was given to students who were merely sitting in and did not cooperate. Also in these courses and seminars, a considerable number of prospective school teachers were active. The state board examination required written theses and very often these theses originated in such advanced courses or seminars and were the nuclei of doctoral theses. Subjects and standards of these state board theses varied greatly. A few examples: Reciprocity theorems in number theory; Morse's relations for stationary points; Morse's theory of conjugate points in the calculus of variations; the role of semicontinuity in the calculus of variations; theory of characteristics for hyperbolic systems of differential equations; characteristic values of partial differential equations in their dependence on the domain; finding explicit formulas for the conformal mapping of given Riemann surfaces on plane domains; critical survey of certain current high school textbooks of mathematics from the viewpoint of mathematical correctness; and so on.

Not infrequently these theses contained some modest original contribution, and in most cases they at least proved that the student had studied some field with real understanding. Sometimes, it is true, results were very poor and unsatisfactory.

At other universities different attempts were made to cope with the mass problem; *e.g.*, at the University of Berlin there was a systematic organization of many smaller study groups of students under the guidance of advanced students and younger instructors. At smaller universities with a much smaller number of students and in the absence of a coherent group of ambitious future scholars, it was easier to maintain personal contact between responsible professors and a large portion of the student body. But almost nowhere was it possible to restore the idyllic pre-war situation, where the whole mathematics group at a university was somewhat like a family with mutual personal confidence and contacts. Where this mathematical "family life" was preserved and even developed after the war, as in Göttingen, it was really restricted to the comparatively small, though by no means exclusive, top group. This class division could not fail to create among the less fortunate majority the psychological atmosphere of an inferiority complex, which so easily becomes dangerous for scientific institutions and which was, in connection with the hopeless outlook regarding jobs, an important factor in the events at the universities in 1933.

While the period between the war and 1933 did not, in spite of honest attempts, solve burning problems of mathematical instruction at the universities, it certainly produced more mathematical progress than most previous periods of similar length.

Since 1933, the total revolution within the system of German universities has so drastically curtailed the number of students that the mass problem of the previous period simply no longer exists. It seems premature to forecast the trend of the future development of mathematics at German universities.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A LINKAGE FOR DESCRIBING CURVES PARALLEL TO THE ELLIPSE

R. C. YATES, University of Maryland

Any point P of a bar AB with its ends moving along two perpendicular lines describes an ellipse. If this bar be joined at its midpoint to the origin (the intersection of the lines) by another bar, the half of the original bar not containing P may be discarded.

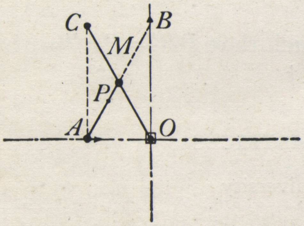

FIG. 1

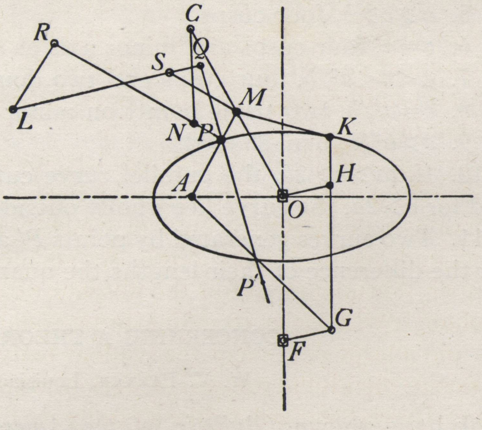

FIG. 2

In any position of the jointed bars the direction of motion of M in Fig. 1 is perpendicular to OM while A has always a horizontal direction. Thus the instantaneous center of rotation of the bar AM is the point C where the vertical line through A meets OM extended. The normal to the ellipse at P then must always pass through C. Note that C is carried around at the extremity of the bar OC which is twice the length OM; also that $OM=MA$.

If we now add two further bars CN and PN such that $MP=NP$ and $MC=NC$ we have only to arrange for another bar attached at P to bisect the angle MPN. This may be done by a special arrangement of two Hart contraparallelograms as shown in Fig. 2.

The linkage $AGKMOH$ was devised by Kempe in 1874. Here $AG=KG=4$; $AM=KM=OM=2$; $KH=OH=FG=1$. If O and F be fixed to the plane so that FG is parallel to OH then A moves along a line at right angles to the fixed line OF. Thus P describes the ellipse. One contraparallelogram $RLQP$ is attached to another $QSMP$ such that $RL=QP=MS=\lambda$; $QL=PR=\mu$; $QS=MP=\nu$; and $\lambda^2=\mu\cdot\nu$.

For this arrangement QP bisects angle NPM and is directed therefore to C. It is in all positions normal to the ellipse and any fixed point of it such as P' describes a parallel curve.

This set of curves is of the eighth order and fourth class and has been studied extensively by Catalan, Cauchy, Breton, Cayley, and others. If we let $MP+MO=a$, $AP=b$ be the semi-axes of the ellipse and c be the parallel distance PP' (where P' is on the "inner" side of the normal) then the curves have the following characteristics:*

1. $c<b^2/a$, oval-shaped.
2. $c=b^2/a$, two triple points on major axis.
3. $b>c>b^2/a$, four real cusps and two double points on major axis.
4. $c=b$, four cusps and contact point at center with major axis as double tangent.
5. $a>c>b$, four cusps.
6. $c=a$, four cusps and minor axis as double tangent.
7. $a<c<a^2/b$, four cusps and two double points on minor axis.
8. $c=a^2/b$, two triple points on minor axis.
9. $c>a^2/b$, oval-shaped.
10. If $2a>c>2b$, the parallel curve cuts the ellipse; if $c=2a$, it is tangent; and for $c>2a$, the curve lies wholly outside.
11. Two curves generated by points P' and P'' such that $PP'=PP''=c$ have for the difference of their lengths the quantity $4\pi c$.†

CONCERNING A THEOREM ON MATRICES

W. V. Parker, Louisiana State University

I have shown (*Bulletin of the American Mathematical Society*, vol. 38, 1932, pp. 259–262.) that if M is a real symmetric matrix of order greater than four in which all principal minors of orders *one* and *four* are zero, then M is of rank three or less. One may naturally inquire if the corresponding theorem is true for Hermitian matrices. The purpose of this note is to show by an example that the theorem is not true for Hermitian matrices in general.

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* See, for example, G. Loria: Ebene Kurven, Teubner, 1902.

† Dienger, Über die Rektification und Quadratur der Toroide, Archiv der Mathematik, vol. 9, 1847.

In the Hermitian matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 1 & 0 & 1-\sqrt{3}\,i & 7+2\sqrt{3}\,i \\ 1 & 1 & 1+\sqrt{3}\,i & 0 & 3+2\sqrt{3}\,i \\ 1 & 4 & 7-2\sqrt{3}\,i & 3-2\sqrt{3}\,i & 0 \end{pmatrix}$$

all principal minors of order four are zero but the matrix is of rank five.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Theory of Equations. By J. M. Thomas. First Edition. New York and London, McGraw-Hill Book Company, 1938. 10+211 pages. $2.00.

Introduction to the Theory of Equations. By Louis Weisner. New York, The Macmillan Company, 1938. 9+188 pages. $2.25.

Einführung in die neueren Methoden der Differentialgeometrie. By J. A. Schouten and D. J. Struik. Zweiter Band: Geometrie von D. J. Struik. Zweite vollständig umgearbeitete Auflage. Groningen-Batavia, P. Noordhoff N.V., 1938. 12+338 pages. Fl. 11.50, geb. Fl. 12.50, RM 16, geb. RM 17.50.

Zur Differentialgeometrie der Kurven und Flächen fester Breite. By H. Buckner. (Schriften der Königsberger Gelehrten Gesellschaft, Naturwissenschaftliche Klasse, vol. 14, no. 1.) Halle, Niemeyer Verlag, 1937. 22 pages. RM 2.

Funktionentafeln mit Formeln und Kurven. By E. Jahnke and F. Emde. Third Revised Edition. Leipzig and Berlin, Teubner, 1938. 12+306 pages. RM 15.

Tests of Significance in Reversal or Switchback Trials. By A. E. Brandt. (Research Bulletin 234, Iowa Agricultural Experiment Station.) Ames, Iowa, Iowa State College, 1938. 27 pages.

Design of Agronomic Experiments for Plots Differentiated in Fertility by Past Treatments. By H. C. Forester. (Research Bulletin 226, Iowa Agricultural Experiment Station.) Ames, Iowa, Iowa State College, 1938. 34 pages.

Descriptive Geometry. By F. A. Smutz and R. F. Gingrich. (Essential Principles and Applications for Students of Engineering and Architecture.) Second Edition. New York, D. Van Nostrand Company, 1938. 10+227 pages. $2.50.

Intermediate Algebra. By W. E. Brooke and H. B. Wilcox. New York, Farrar and Rinehart, Inc., 1938. 8+323 pages. $1.90.

Teaching Arithmetic in the Elementary School. By Robert Lee Morton. Vol II, Intermediate Grades. New York, Silver Burdett Company, 1938. 12+538 pages. $2.72.

Analytic Geometry. By Clyde E. Love. Third Edition. New York, The Macmillan Company, 1938. 14+279 pages. $2.75.

Solid Mensuration. By W. F. Kern and J. R. Bland. Second Edition. (With proofs.) New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1938. 7+172 pages. $2.00.

A Short Course in Trigonometry. By J. G. Hardy. Revised Edition. 9+152 pages. *Logarithmic and Trigonometric Tables*. Revised Edition. (Prepared under the direction of E. R. Hedrick. Entirely re-set in a new type face.) 16+143 pages. New York, The Macmillan Company, 1938. $2.25.

Advanced Analytic Geometry. By A. D. Campbell. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1938. 10+310 pages. $4.00.

The Ringstrom Three Dimensional System of the Atoms (*With Atomic Musical Pitches*) *and the Theory of Atomic Integration*. By A. B. Ringstrom. Chicago, Vega Publishing Company, 1938. 87 pages.

A Preface to Mathematics. By C. E. Van Horn. Boston, Chapman and Grimes, Inc., 1938. 12+124 pages.

The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry. By Nathan Lazar. New York, The Mathematics Teacher, 1938. 66 pages. $1.00.

Tafels van e^x. By H. W. Holtappel, Samengesteld door. Groningen-Batavia, P. Noordhoff N.V., 1938. 31+132 pages. fr. 6.

Approximate Computation. By Aaron Bakst. (The National Council of Teachers of Mathematics, the Twelfth Yearbook.) New York, Columbia University Teachers College Bureau of Publications, 1937. 16+287 pages. $1.75.

Some Notes on Least Squares. By W. E. Deming. Washington, D.C., The Graduate School, The Department of Agriculture, 1938. 4+181 pages. $1.50.

REVIEWS

Urphänomene der Geometrie. Erster Teil. By Louis Locher. Zürich-Leipzig, Orell Füssli Verlag, 1937. 16+164 pp. R.M. 3.60.

A few years ago, a major prophet among American mathematicians—whether seeing visions with the young men or dreaming dreams with the old is not quite clear—pronounced a Jeremiad entitled *The rise and fall of projective geometry*.* During the past decade, however, numerous books have appeared on the subject both here and abroad, among which indeed might properly be counted the prophet's own treatise on algebraic curves. Comes now the author of the little book under review, a purely synthetic treatment of the elements of projective geometry, designed for the wider circle of lovers of geometry, to declare his conviction that "More people than one might think yearn for a kind of activity that guarantees that man's essential being participates in a purely intellectual world." His judgment seems justified by the inclusion of *Euclid's Elements* in the Everyman series, that *rara avis* a book on mathematics among the best sellers, not to mention the success of the popular books of E. T. Bell and Dantzig and the Liebers' modernistic versions of classical theories. The author feels that the *Elements of Euclid* may justly be regarded as a significant civilizing influence, which has become a part of the cultural inheritance of the race. He believes that no less may be expected of projective geometry, which, he laments, is familiar to a very narrow circle of specialists. Were such ideas as the principal of duality thoroughly assimilated he thinks that projective geometry might "suggest to the natural scientist entirely new types of conceptual patterns."

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* This MONTHLY, vol. 41, 1934, p. 217.

A word about the arresting title of the book may be appropriate. The author decries the abstract formulation of mathematics when carried so far that it loses its connection with reality. To emphasize his view he refers to the properties of space as *phänomene*, a word which with the prefix *ur*, signifying primitive or original, makes up the term which seems to be new to geometry. He says in the preface: "The word axiom will be avoided in the following—it smacks strongly today of a wholly arbitrary, capricious human invention. Since for me the mathematical concept-structure is not a mere figment of the mind, but represents an abstract echo of a genuine world of a physico-mental origin, it seems to me that *Urphänomen*, the word introduced into color theory by Goethe, is more expressive, more suggestive of reality." Thus the word is not used as a mere substitute for axiom, although he lists as *Urphänomene* numerous propositions, together with their duals, which might ordinarily be included in a set of axioms. Again after exhibiting the Möbius band as a one-sided but bounded surface, he presents the projective plane as one-sided but unbounded. He considers a sphere of constantly increasing radius which is metamorphosed into the plane at infinity, one side of the sphere (the outside) completely covering the plane and concludes: "This metamorphosis (*grenzübergang*) of the sphere into the plane at infinity represents the *Urphänomen* for the one-sidedness of the plane."

While borrowing from Goethe, he also appropriates another word, polarity, which he uses throughout in place of the usual duality or reciprocity. This seems to me unfortunate both because the terms replaced are well established and satisfactory, and furthermore polarity is already used in geometry in another sense.

The first chapter deals chiefly with Desargues's theorem on perspective triangles in the plane and the related 10_3, 10_3 configuration. The theorem is proved only experimentally, illustrated by numerous figures.

In Chapter 2 are introduced the three geometric elements, point, line, and plane, which in combination yield the seven primitive forms.

Chapter 3 is concerned with "The Phenomena of Connection," *i.e.*, junction and intersection of the elements and primitive forms in three dimensional space. Axioms (*Urphänomene*) of existence, of uniqueness and some combining both ideas under the "one and only" terminology are presented in parallel columns, leading up to a formal enunciation of the principle of duality, already foreshadowed in the previous chapter.

The longest chapter of the book—about one fourth of the whole—is devoted to "The Infinitely Distant in Geometry." He discusses at length and with profuse illustration the surprising effects on figures of two major projective phenomena—that the line is unbounded and closed, the plane unbounded and one-sided. Much space is given to the regions into which sets of lines and planes respectively divide point spaces of two and three dimensions. The infinite elements act as unifying agents, sometimes reducing, sometimes increasing the cases to be considered. Thus three lines divide the projective plane into only four regions, an instance of simplification. But we find the opposite effect when enumerating the "triangle-figures" which are determined by three points. In elementary geometry we have a single figure, consisting of the three finite segments joining the points, whereas here we get eight—each of the others omitting at least one of the finite segments.

The fifth chapter takes up the simpler complete polygons and polyhedrons, culminating in the proof of Desargues's triangle theorems by means of the complete 5-point in space.

Chapter 6 considers the planar Möbius net—and the space analogue—the configuration arising from the quadrangle-quadrilateral configuration which finds its metrical version in any one of the several lattices. It concludes with a proof of the fundamental theorem of projective geometry, the Pappus-Pascal theorem on triply perspective triangles and a property of the quadrangle published as recently as 1935.

It seems odd that a book which omits the general Pascal theorem should include a discussion of Cassinian ovals, which occupies the major part of the final chapter. But it furnishes at once a tie with the first chapter and the basis for some concluding philosophical remarks. Such observations, as well as comments on the beauty, significance, and novelty of the results, which are interspersed throughout the book seem appropriate in a work of this kind in which the author plays the rôle of guide and companion no less than master.

On the whole the book is well done. I noticed only two trivial misprints: Fig. 125 is labeled 521, while Fig. 170 is cited in the text as 169. It is abundantly illustrated, the number of figures exceeding the number of pages. The figures moreover are exceptionally well executed, clearly and accurately drawn on pure white paper, which is free from glare. The esthetic qualities of the book set a high standard—the printing is clear, the binding attractive. The book is comparable in size and purpose to the Carus Monographs, which however in my judgment yield to this in general artistic merit.

R. M. WINGER

A Course of Pure Mathematics. By G. H. Hardy. Seventh Edition. Cambridge, The University Press; New York, The Macmillan Co. 12+498 pages. $3.75.

This is the seventh edition of a book first published in 1908. The subject matter, fundamental in analysis, is largely that of the elementary calculus. The thorough competence with which the author gives discussions and definitions, and the sensible rigor with which he gives proofs, have made this book one of the "best sellers" among mathematics books. It is lamentable that this book is "too hard" for a text in an ordinary first course in calculus. For a superior student in mathematics, it should be more intelligible and inspiring than is the conventional mediocre elementary calculus text based on a definition of limit so vague as to make a researcher in analysis shudder at the suggestion that he use the definition of limit to prove that if $\lim_{x\to 0} f(x) = a$ and $\lim_{x\to 0} g(x) = b$, then $\lim_{x\to 0}[f(x)+g(x)] = a+b$.

The new edition contains a few new sections, some sections which have been rewritten, and numerous additional problems of which many are recent Tripos examination questions. These Tripos problems are sufficiently difficult to sharpen the wits of a superior undergraduate student. Errors and misprints are almost totally absent from the book. One oversight which persists through the various editions can be corrected by interchanging "negative" and "positive" in the proof of Theorem D of section 124.

R. P. AGNEW

La Cryptarithmie, Extrait des Comptes Rendus. By M. Pigeolet, Brussels, Librairie du Sphinx, 1935. 16 pages. 1 belga.

The author defines Cryptarithmie as the decoding of numerical computations in which letter symbols have been substituted for various digits. He then briefly outlines the history of this subject, classifies different types of such problems and indicates methods of attack. Bases for classification are: whether or not one or more of the digits remain uncoded; whether the distinct digits are represented by distinct or identical letters; whether the computations involved are those of addition, subtraction, multiplication, division, extraction of roots, determination of greatest common divisors, *etc.*; whether the basis of enumeration is ten or some non-decimal base; and whether or not the problem is dependent on primes, powers, logarithms, or other functions.

While M. Pigeolet solves some of the many problems he cites, he leaves enough unsolved to fascinate the average reader for several hours. He concludes by pointing out that there is a spiritual satisfaction to be had from dealing with pure numerical relationships and wresting truth out of intangibles.

Although the name, Cryptarithmie, is but seven years old, many representative problems in this field are somewhat older. Several appeared in the MONTHLY in 1921, and are cited by W. W. R. Ball in the tenth edition of his *Mathematical Recreations and Essays*, pages 248–253.

W. F. CHENEY, JR.

Statistical Methods. By G. W. Snedecor. Ames, Iowa, The Collegiate Press, 1937. 13+341 pages. $3.75.

The mathematician who opens this book with the expectation of being entertained in mathematics will be greeted with disappointment; but if he is desirous of learning how statistics is being used today, at least in some quarters, he will have found a gem. The work is a lucid and delightful exposition of R. A. Fisher's developments. Intricate notions associated with experimental design are very cleverly explained. The book deserves widespread adoption as a text. It can be used in classes where the mathematical attainments are little or nothing beyond college algebra; yet there is plenty of material for a year's course for more matured students, in which event supplementary reading in the way of derivations should be provided.

Statistics, whatever the word originally meant, is today the symbol for thought regarding observed phenomena; what can be inferred from an experiment, and how will further data behave? All branches of science meet in statistics. No one has ever drawn a conclusion empirically without using statistical methods in one form or another, good or bad. As Mr. Snedecor says in his preface, "Statistics is a mode of thought." It is only natural that a subject so vast should present itself in different aspects to different people; Mr. Snedecor's treatment is to be regarded as an exposition of the methods that he himself has put to use at Ames. The same methods in the hands of anyone but a natural scientist of experience can do great harm. The reader will be impressed throughout by the fact that the author is not interested primarily in the subject for itself, but in the fundamental problems for which statistical calculations may be made. Illustrative examples abound, with clear and easily followed directions for machine calculation—enough detail, yet not too much.

The language of statisticians remains ambiguous, owing to a failure to differentiate between what is mathematics and what is the actual behavior of nature. For instance, on page 51 we read, "It is from this that we must estimate the population standard error. . . . In practice, the experiment is seldom repeated," and then on page 55, "The sample value far exceeds even the 1% level, 2.977, and this latter would be expected to occur only once in 100 samples if m were really zero." If experiments are not usually repeated, how can such statements be confirmed? It can not be denied that there is something useful in the calculations on which such statements are based, but only so for the man who first of all knows his wheat, pigs, fertilizers, or electrons: others must expect to be misled.

In the table of chi-squares on page 10, the author might have pointed out that with 20 items, chi-square can take only the 11 values 0, .2, .8, 1.8, $\cdots$, 16.2, 20; the discontinuous nature of chi-square, not chance, is responsible for the empty cell of class 1–1.49, as it is also elsewhere. In the treatment of least squares, the author fails to point out that in his formulas the error is assumed to be all in the y coördinate.

In the way of other minor weaknesses, the reviewer would express regret at the lack of perspective that the book may create: not that what the author says could be told in less than 341 pages, but that the reader could now and then have been furnished references to other sides and arguments, or at least warned that this is not all. There is no mention, for example, of any criterion for the choice of test that a person should use. Closely associated with such a criticism is the fact that what little history is given is in most instances unreliable: the first calculation of a pair of confidence or fiducial limits was made by E. B. Wilson in 1927, rather than by Fisher in 1930 (p. 60); the analysis of variance (p. 180), not in name but in thought, goes back at least to 1912, as is evident from pages 66–71 of Palmer's *Theory of Measurements* (1912). Indeed, the idea goes back to the introduction of the correlation ratio by Karl Pearson in 1905. On page 30 Mr. Snedecor attributes the factor $n/(n-1)$ to Student in 1908, and on page 51 he apparently attributes it to Bessel, no mention being made of the fact that Gauss in 1823 very clearly subtracted the number of unknowns evaluated from the number of observations to get the corrected denominator (number of "degrees of freedom") in an estimate of the population variance. It was Helmert in 1876 who first derived the distribution of s^2; Student, so far as I can see, was not concerned in 1908 about the average value of his s^2, and made no mention of $n-1$.

Many people have been favorably impressed with the attractive cover; the paper and printing also do credit to the publisher, except that the setting of some of the mathematical formulas does not always do justice to the author's genuine and usually successful attempt at clarity.

W. E. Deming

Calculus. By Edward S. Smith, Meyer Salkover, and Howard K. Justice. New York, John Wiley and Sons, 1938. 12+558 pages. $3.25.

This text is not just another book on calculus; in some respects it is unique. For instance, the order of arrangement of the topics departs not a little from the traditional scheme of two parts, *viz.*, differential and integral. The reader is introduced to both phases of the subject early in order that he may appreciate it as a science, a philosophy, and not just a body of formulas and rules. In the first three chapters there is developed a general notion of the calculus, using only polynomial functions. Sufficient rigor is employed, however, to establish sound foundations but not a discouraging amount for the beginner. All of the definitions are couched in language tending to impart to the reader (beginner) a rational perspective of the science. The first chapter is written in a form to allure the student and yet to challenge his thinking.

And I suspect that the authors have purposely avoided the title "Four Step Rule" in arriving at the ratio $\Delta y/\Delta x$ on page 15, since the aim is to put in the background the notion of "formulas and rules." Perhaps not too much space has been devoted to the delta ratio in chapter II. At any rate if the student does all of the problems by the delta ratio which are listed, he will have mastered the technique of the process. The completion of chapter III should furnish the student with a fair notion of the meaning and use of integral calculus. Chapters IV to X, inclusive, apply the principles learned in chapters I to III, but using now algebraic and transcendental functions. The student is employing in these chapters the calculus and is learning more of its outreach. The analysis and rigor exhibited in article 51, "The Limit Defining *e*," is a fitting illustration of the scientific study of a question and yet is not beyond the ability of a student who can successfully reach this point in the text. The discussion is not found in very many texts, but the reader will find it challenging and stimulating. Many of the more difficult topics are treated toward the end of the book. Indeterminate forms are first encountered in chapter XII, infinite series in chapter XIII, and expansion of functions in chapter XIV. The remaining three chapters treat of hyperbolic functions, partial differentiation, and multiple integrals.

The authors have made a very reasonable and lucid approach to the calculus, introducing special devices occasionally which serve to enkindle a keener interest in the subject; see page 259, article 92, as an illustration. The drawings are well executed, a table of integrals is found at the end, and there are answers to all the problems.

RICHARD MORRIS

Nouvelle Méthode pour construire et dénombrer certains Carrés magiques d'ordre 4m avec application aux Parcours magiques, Extrait du Sphinx. By Dr. Erich Stern. Brussels, Librairie du Sphinx, 1937. 20 pages. 1 belga.

This treatise, translated from the German into French by General E. Cazalas of Versailles, proposes and explains methods by which the 64 numbers from 0 through 63 may be arranged in a magic square in more than 8×10^{20} essentially different ways. The author proceeds to generalize his result to apply to any magic square of order $4m$.

The method consists in the formation of two auxiliary squares, U and V, such that the numbers in one quadrant of each square may be arranged arbitrarily (except for minor restrictions), with the other quadrants of each square formed from the arbitrary quadrant by certain rules of symmetry. Then these two squares, U and V, are combined in a simple linear relationship to form a magic square.

The author determines the number of ways in which a quadrant of a square may be filled out consistently with the rules he lays down, and the number of variations which may be introduced through changing the linear relationship for combining U and V, and by altering the type of symmetry between the quadrants of each square, and thus arrives at the prodigious total of 800,708,749,625,917,440,000 essentially different, 8×8, magic squares. He sketches an extension of his methods to apply to larger magic squares of order $4m$, and concludes by pointing out untouched possibilities attainable through the pursuit of these methods for magic squares of *any* order.

W. F. CHENEY, JR.

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N.J.

CLUB TOPICS

The following play was written and produced by Professor A. Marie Whelan of Hunter College of the City of New York. It would seem that many clubs might be interested in including it in their year's program, hence its publication at this time.

It Can't Happen Here

A mathematical musical farce

Time: *The present.*
Scenes: *All action takes place in a mathematics class-room.*
Note: *Professor Math slips from his rôle of Professor into the rôle of Pure Mathematics, to suit his convenience, and without warning or apology.*

Prologue

(*Prof. Math strides onto the stage. He wears a crown, and sings the following song. Tune: I'm Called Little Buttercup.*)

Math:
I'm King Mathematics, Engaged in dramatics.
Your pardon I humbly request.
Your patience I ask for;
Don't take me to task for
A deed that I do by request.
I've strayed from my limits,
And for these few minutes,
I fear I am not at my best.
My rôle is not writing, or acting, or fighting.
My kingdom's a haven of rest.
Through infinite spaces, Through far distant places,
On wings of Logic I climb.
No body to stay me, No object to sway me.
My kingdom's the reach of the Mind.

(*Exit*)

ACT I.

Scene I. (*The Mathematics class is gradually assembling. The last bell has not yet rung. One student is sitting at a desk, looking dejectedly at a test paper. A second student approaches her.*)

Helen: What is the matter, Jane? You look like your last hope is gone.

Jane: So it is! Look at that, Helen! Another "F"! I don't know what I am going to do.

Helen: Don't let it discourage you too much. I understand that practically everybody flunked this time. It was a pretty hard test.

Jane: *All* tests are hard for me. I have taken this course three times already, and I don't know any more about it *now* than I did the first time I took it. I am a graduating senior, but I won't graduate if I don't pass this course. And

I haven't a Chinaman's chance. But what is the use of talking to you? You'll never understand. I suppose you got "A" as usual?

Helen: Yes, I did.

Jane: I don't know how you do it.

Helen: Perhaps I work harder than you do. Why don't you try working a little harder?

Jane: It wouldn't do me a bit of good—and I don't believe in wasting my work. I like proper return for my labor, thank you. Once in a blue moon, I think I have caught on to something—but when I try to use it, it seems that it doesn't apply. I gave up long ago. I think it is all nonsense anyway. What do you see in it? Why do you work so hard?

Helen: I don't mind the work. I love it.

Helen sings: (*Tune: Tit Willow.*)

Of work in Math'matics, I'll never complain.
For I love it! I love it! I love it!
And the thrill that it gives me, I'll never explain.
But I love it! I love it! I love it!
If a problem is difficult, all the more fun,
And the pleasure I get, when at last it is done.
OH, the thrill that I feel, when the battle is won!
I love it! I love it! I love it!

Jane:

From work in Math'matics, I'll always refrain,
For I hate it! I hate it! I hate it!
Of the pain that it gives me, I'll ever complain.
And I hate it! I hate it! I hate it!
And the silly old battles of A, B, and C!
Does it matter who wins, When the loser is ME?
OH, the thrill that I'll feel,
When at last I am free!
I hate it! I hate it! I hate it!

Chorus: (*The entire class repeats Jane's song.*)

Jane: Well, Helen, it seems that the majority opinion is with me. (*Reflectively.*) Majority Opinion. Say, that gives me an idea!

(*The bell rings. The students hurry to their seats, as Professor Math enters the room.*)

Prof. Math: As usual, I must take the attendance. Will you please sign your names on the slip of paper, that I shall pass around?

(*Prof. Math hands a slip of yellow paper to a student in the first row. The student signs it and passes it along. In the meantime, Jane cautiously slips a similar piece of paper to her neighbor. Both papers are seen to pass from student to student, until Jane's paper reaches the front row. Prof. Math, thinking that it is the attendance slip takes it from a student, who is obviously reluctant to let him have it. The words on it catch his attention and he reads them aloud.*)

Prof. Math: A meeting is called of the students in this class to discuss a plan to ensure that everyone in the class shall pass the course.

Room: 1-125 Time: 12:15

DON'T FAIL TO COME!!!!!

(*Addressing the class*) Very interesting, indeed! No signature, unfortunately. Who wrote this notice? (*No one answers.*) Come now, don't be backward. My only reason for wanting to know is to congratulate the author. I am delighted to find out that someone is taking this serious problem seriously. I assure you that nothing would give me greater pleasure than to pass every student in this class. But I must admit that the probability of that happening is very small—considerably less than one-twentieth, in fact. (*Pauses a few seconds.*) Well, it seems that my curiosity is not to be satisfied. So let us get down to work. Judging from the last test, I have decided that a little review of elementary algebra is in order. Mary, will you go to the board, and solve this equation;

$$x^2 - 2x = 0.$$

(*Mary goes to the board and solves the equation, using the Quadratic Formula.*)

Prof. Math: Why didn't you solve it by factoring? Here, let me show you.

(*Prof. Math solves equation by factoring.*)

Mary: Those are the same answers I got, Professor.

Prof. Math: I did not say that your solution was *wrong*, Mary. See here. Suppose that you wanted to go some place, and that you could get there by bus or subway train. Suppose also that you would have to wait one half-hour for the bus, but could get the subway train immediately. What would you do?

Mary (*innocently*): I would wait for the bus, Professor. I don't like the subway.

(*General laughter.*)

Prof. Math (*joining in the laughter*): Well, you certainly flattened my point! Incidentally no wonder you are late so often. Anne, go to the board, now, and solve this equation:

$$x^2 - 2x - 2 = 0.$$

(*Anne presents the following solution:*)

$$\begin{aligned} x^2 - 2x - 2 &= 0 \\ x^2 - 2x &= 2 \\ x(x - 2) &= 2 \\ x &= 2 \\ x - 2 &= 2 \\ x &= 4 \end{aligned}$$

Prof. Math (*sarcastically*): I suppose *you* never heard of the Quadratic Formula?

Anne: I *do* know the Quadratic Formula, Professor. (*Rattles the formula, parrot-like.*) That is right, isn't it?

Prof. Math: Quite right! May I ask why you did not use it?

Anne: I thought you didn't *like* it. So I solved it exactly the way you did *that* one. (*Pointing to the preceding problem.*) I only wanted to please you.

Prof. Math: Very obliging of you, I'm sure. For your information, you did *not* please me. Far from it! There ought to be a law against that kind of algebra. What am I saying? There is a law against it. You look like a failure to me. I've certainly got *you* on my list.

Prof. Math sings: (*Tune: They'll None of 'Em Be Missed.*)

Since someday it will happen that my failures must be found,
I've got a little list! I've got a little list!
Of Math'matical offenders, that forever do abound—
And who never will be missed.

Who never will be missed.
There's the ever present nuisances, who dare be late for class
All those who murder Algebra, and cannot draw their graphs.
All students who neglect their work, and cut my quizzes flat.
All students who, in finding roots, find roots, like *you*, like *that!*
All students who use standard forms, where standard forms don't fit.
They'd none of them be missed! They'd none of them be missed!

Chorus: He's got us on his list! He's got us on his list!
And we'll none of us be missed! We'll none of us be missed!

(*Curtain*)

Scene II. (*Time: 12:15. Several students are discussing the proposed meeting.*)

Ruth: Who called this meeting, anyway? Do you know, Anne?

Anne: I don't really *know*—but I think it was Jane. The paper seemed to start from her direction. It was stupid of you, Mary, to let Prof. Math get his hands on it.

Mary: Go ahead and blame me. What else could I do? I bet you wouldn't have done any better. Anyway no harm is done. He did not find out who wrote it.

Anne: No, he didn't. It might have been a bit exciting if he had found out that Jane started it—that is—if she did start it. Here she comes now. Let's ask her.

Ruth: Hello, Jane. Are you the one who called this meeting?

Jane: Yes, I did. But I don't want Prof. Math to find out that I did. I do think you might have been more careful, Mary.

Mary: I tell you that I couldn't help it. Anyway, I don't see why you didn't admit writing it. He said he only wanted to congratulate you. (*Laughter.*)

Ruth: You *are* a simpleton, Mary! Congratulate her indeed! He'd have taken her apart. What is it all about, anyway, Jane? I am having a Physics test, next hour, and I really should be studying now—but my curiosity got the best of me. I warn you. It had better be good. Get on with it—everybody is here, I think.

(*Jane goes to the desk, and addresses the meeting.*)

Jane: The meeting will please come to order! As you see, I have taken it upon myself to call this meeting. You shall know my reasons. I understand that nearly everybody in this class flunked the last test in Mathematics. I want the exact figures. How many got "F" on the last test? Let's have a show of hands.

(*Everyone, except Helen, raises her hand.*)

Jane: It is worse than I thought!

Ruth: I don't think I agree with you. I think the fact that so many flunked is a good sign. After all, he can't very well fail the whole class.

Jane: Don't kid yourself! You don't know him! He'd fail us all without a qualm, just to uphold his silly old standards. I tell you the case is desperate. And desperate cases require desperate remedies. I know exactly what the trouble is. Prof. Math has too much power! Take my case, for instance. He has the power, and he'll undoubtedly use it, to fail me in this course.

And what is the result? I don't graduate.—All my plans for the future are destroyed. My whole life, perhaps, is ruined. And all for what? To uphold his inhuman standards. I ask you: Is it *right?* Should any one person have that much power? Again, I ask: *Is it right?*

Cries from all directions: NO! NO! NO! It isn't right!

Anne: But what are you doing to do about it?

Jane: I'll tell you. This autocratic procedure must be abolished! It is upheld by ancient custom, *that,* I know—But TIME MARCHES ON!—And I tell you the time has come for the students to take into their own hands the power to regulate their own assignments, and to determine the mark they shall receive for their work. Time does march on! And the time is HERE—TO-DAY—To abolish the AUTOCRAT OF THE CLASS-ROOM! (*Cheers.*)

Jane: And what shall we substitute for him? I'll tell you—The answer is—MAJORITY RULE! (*More cheers.*) And how shall we establish Majority Rule? I'll tell you that too—We'll form a—UNION!

Excited cries: Yes, yes, let's form a Union!

Jane (*continues*)*:* Yes, we will form a Union. I'll be the leader—After all it is my idea and I'll draw up a contract. With you 100% behind me, Prof. Math will be compelled to sign it. But I must have *100% support!* The contract will specify that our assignment shall consist of only one problem a day; and that the grades in the course shall be determined by a Majority Vote of all the students in the class. Of course (*laughing*) by mutual agreement among ourselves, that will mean that we will all get "A."

Anne: I think it is a wonderful plan—but if we can go that far, why not go a little farther, and abolish assignments, altogether?

Jane (*with absolute sincerity*)*:* Don't you think that would be a little one sided?

Anne: Perhaps it would—after all a contract does imply something mutual.

Ruth: Jane, you said that you would have to get 100% support. I don't think you'll get it. Helen, there, won't join our Union, I'm sure of that.

Jane (*grimly*)*:* She won't, eh? We'll see about that. (*Marches up to Helen, and grips her firmly by the shoulder.*) How about it, Helen? Will you come in with us?

Helen: Of course I won't. Why in the world should I? What would I get out of it? I always get an "A" anyway—an honest "A." And as far as the work is concerned, well I like to work.

Jane: Do you hear that, Girls? Nice public-spirited citizen, she is! She likes to work, and as long as she gets her "A," what does she care if we all flunk—Well, we have a name for her kind, haven't we? (*Hisses and boos.*)

Mary: I do think you are a little selfish, Helen.

Ruth: Selfish?—That is too mild a word for her. She is a Public Enemy! Public Enemy Number One! (*More boos.*)

Anne: I'll tell you what she is—She is a (*Pausing*) REACTIONARY!

(*The crowd takes up the word "Reactionary," dinning it into Helen's ears. Helen shows signs of weakening.*)

Jane: I'll tell you what she really is—an INTELLECTUAL ROYALIST! (*Boos and hisses.*)

(*Jane chants dramatically, always with menacing gestures toward Helen.*)

Jane: She's an intellectual royalist,

A mathematical loyalist,
She'd block us if she could.

Anne (*taking up the chant*):
Selfish-hearted adversary,
Dyed-in-the-wool Reactionary
Opposed to the public good.
Opposed to the public good.

Chorus (*repeats above adding*):
Down with all Reactionaries,
Down with all our adversaries,
Opposed to the public good!
Opposed to the public good!

Helen (*tearfully*): OH, STOP! Please STOP! Don't hurl those dreadful words at me. Those words, those awful words, I can't stand up against them. I give in—I'll join your Union.—What do you want me to do?

Jane: Exactly what you are told. I am the Leader.—You back me up in every action.—Well, Girls, now that we've taken care of Helen, we are ready for a vote.—All in favor of forming a Union, say "AYE." (*All say "AYE."*) Opposed? (*None.*) The motion is carried. Well the first step is accomplished—very successfully accomplished too—100% Unionized! Girls, we're going places!

Song by Jane: (*Tune: Funiculi, Funicula.*)

Jane: I think it well indeed, to form a Union.
Chorus: And so do I, and so do I.
Jane: I think our failure lies in our disunion.
Chorus: And so do I, and so do I.
Jane: And why, why should I spend my time in slaving,
The whole day long, the whole day long?
To strike a valiant blow for Student Freedom.
Is far from wrong!
Chorus: Is far from wrong!
Hearken, Hearken, Union wins the day.
Hearken, Hearken, Union points the way,
To get an "A," to get an "A," to get an "A," to get an "A"!
Union points the way, we'll Unionize, we'll Unionize!

(*Curtain*)

ACT II

Scene I. (*Professor Math is standing at his desk, considering the contract which has been presented to him. The students are lined up behind Jane, who is standing to the left of the desk.*)

Jane: Sign the contract, Professor Math. You have no other choice. You have got to sign the contract!

Chorus: (*Tune: For He's Going to Marry Yum-Yum.*)
You have got to sign the contract!
Resistance is idle, your anger, pray bridle.
I think you had better succumb-cumb-cumb,
And sign on the dotted line.

In this matter, we'll brook no delay, delay.
It's no use to fight it, for now we're united,
And bound to have our way, our way.
You've got a good bargain at that.
In this matter, we pray you admit, admit;
There's nothing to do but submit, submit.
For now we're united, there's no use to fight it,
There's no use to fight it. And you've got a good bargain at that.
For we'll give you one vote in the class.
We'll give you one vote, one vote in the class.
We'll give you one vote, one vote in the class.
In the class, in the class, in the class, in the class!

(*Prof. Math signs the contract, and hands it to Jane.*)

Prof. Math: Here is your contract, signed on the dotted line. It seems I had no other choice. I bow to your greater numbers. Let me congratulate you on the the success of your plan—that is, on its apparent success. You must forgive me, if I quote some old proverbs; "Things are not always what they seem" and "He who laughs last, laughs best." I warn you that your Union is not going to accomplish all that you hope for—Now, don't misunderstand me. I am not opposed to Unions. Far from it. For some problems, they seem to be the only solution. I might say that they are a standard form of solution—but, like all standard forms, they don't fit all cases. Incidentally, that is something that I have tried to teach you—not very successfully, I confess. Well, Experience is said to be a very effective teacher. In the meantime, if it is not opposed to the will of the Majority, perhaps you will be seated. I understand that you have done one problem for today.—Jane, will you be kind enough to put your solution on the board?

(*Jane goes to the blackboard and presents a solution that is obviously wrong. Her errors compensate, however, and she obtains the correct answer.*)

Jane: My problem is to solve the system of equations:

$$\frac{y-1}{x-2}=\frac{3}{5} \qquad (1) \qquad\qquad \frac{y+1}{x-5}=\frac{5}{2} \qquad (2)$$

I did not *need* the second one. Using the first

$$\frac{y-1}{x-2}=\frac{3}{5}.$$

First "*you say,*" $\frac{y}{x}-\frac{1}{2}=\frac{3}{5}$;

then "*you say,*" $\frac{y}{x}=\frac{3}{5}+\frac{1}{2}$; $\frac{y}{x}=\frac{3+1}{5+2}$;

then "*you say*": $y=4, \quad x=7.$

Prof. Math: Well, the answer is right, to be sure, but that is the only thing right about it. I never saw such a ridiculous piece of Algebra in all my life! You certainly get zero for that recitation!

(*Prof. Math picks up his roll-book. Jane stops him as he is about to record a grade.*)

Jane: Just a minute, Professor—You forget our contract. I call for a vote. How about it, girls? Was my problem right or wrong?

(*Chorus of cries*)*:* Right! Right! Entirely Right!

(*Jane takes the roll-book from Prof. Math's hands and records her own grade.*)

Prof. Math: (*Picking up the book and reading aloud.*) 100%! Yes, (*violently*) 100% WRONG! AH, Truly, I am deposed!

(*Sings: Tune: A Wandering Minstrel.*)

Unhappy Monarch, I. I bow my head in sorrow.
I never dreamed a morrow, would see me to students submit.
They solve all problems wrong, through every error ranging,
My noble laws they're changing, to please their feeble wit.
To please their feeble wit.

(*Sits down and drops his head on the desk.*)

Chorus: Unhappy Monarch, Aye! He bows his head in sorrow,
He never dreamed a morrow, would see him to students submit.
We solve all problems wrong, through every error ranging.
His noble laws we're changing, to please our nimble wit,
To please our nimble wit.

(*Curtain*)

Scene II. (*Students are standing in groups about the room. Others are seated, apparently working.*)

Jane: Well, Girls—Today is the BIG DAY! Think of it, we'll all get "A"! You can thank me for that.

Mary: I can't think of anything else. I feel a bit nervous, to tell you the truth,—like I was standing on a volcano that might blow up any minute.

Jane: Nothing is going to blow up here. Not a chance of it! Don't worry. Everything is under control.

(*The bell rings. Everyone is seated as Prof. Math enters the room.*)

Prof. Math: As today is the first school day of the month, the first thing on the program is the distribution of grades. As I call your names, will you please come forward to receive your mark for last month's work. To relieve your suspense—I will tell you in advance that every student will receive an "A." It may seem a bit ungracious not to offer congratulations on this remarkable achievement, but I offer none.

(*Prof. Math calls each student by name, and presents each with a large cardboard "A". Each student holds up her "A" so that the audience can see it. On the back of the cardboard is a large "F" not visible to the audience.*)

Jane: Come on, Girls, I think this calls for a celebration! Let's sing our song!

Chorus: I thought it well indeed to form a Union.
And so did I, and so did I.
I thought our failure lay in our disunion.
And so did I, and so did I.
For why, why should we spend our time in slaving,
The whole day long, the whole day long?
To strike a valiant blow for Student Freedom,
Is far from wrong, is far from wrong.
Hearken, Hearken, Union won the day
Hearken, Hearken, Union showed the way,

We got an "A," we got an "A," we got an "A," we got an "A"!
Union showed the way, we got an "A," we got an "A"!

(*While the students are singing, Prof. Math goes to the board, and writes in large letters:* $Y=X-100$. *As the students sing the last phrase, they hold up their cards, which they have cautiously reversed, so that the large "F" is now visible to the audience.*)

Mary: Anne! Anne! you *didn't* get "A". You got an "F"!

Anne: It *is* an "F"! You got an "F" yourself! Look, look! So did Jane—Everybody got an "F." Even Helen got "F"!

Jane: We'll see about this! He's let us down! Are we going to let him get away with this? He has broken his signed contract!

Ruth: But he didn't break the contract—He gave us all "A." I know he did.

Anne: Ruth is right. He certainly gave us all "A." That is all he contracted for.—We've got to be fair!—But, I would like to know what happened to our "A's"?

Helen: If you would really like to know—I'll tell you. He worked a transformation. I saw him do it.

Prof. Math: (*Tune: Henceforth All the Crimes.*)

It still remains true, that whatever you do,
The record is bound to reveal it.
Unite, if you must, but don't put your trust,
In Union's might to conceal it.
If you really want "A"—There is only one way
You can get it and part with it never.
For the record won't lie, though you try till you die.
And Truth is triumphant, forever!
No, the record won't lie, though you try till you die.
And Truth is triumphant forever!

(*Prof. Math gets his old crown from the desk and puts it on his head.*)

Prof. Math (*holding up the contract*)*:* I now declare this contract abrogated—and tear up this scrap of paper. (*Pauses a few seconds.*) The class is dismissed for today. You are not prepared to get any value out of it. AND—The assignment for tomorrow is—(*Another pause*) Page 125, the first FIFTY problems! (*Groans from the class.*)

(*Curtain*)

Scene III. (*Jane and Helen are talking. The rest of the class is seated.*)

Helen: Nice mess you got us into! I hope you are feeling proud of yourself! First "F" I ever got in my life!

Jane: It will do you good.—Give you some understanding about the way the other half lives (*laughs*). Other half? I mean the other nine-tenths.

Helen: Well, I warn you, if you have any other crazy schemes to try, you can count me out, right from the start. Once is enough for me, more than enough. The next time I'll have more backbone. I'll keep in mind that old truism: "Sticks and stones may break my bones, but words can never hurt me."

Jane: Oh, yes? That may be old, all right. But it certainly is no truism. Whoever wrote it, did not know much, if you ask me. Words hurt all right—worse than sticks and stones—They hurt you right down where you live!

Helen: Hurt, or not, they are not going to make me participate in any more of your wild plans. If you'll take my advice, you will settle down to work in earnest. Prof. Math is too smart for you. You might as well admit it.

Jane: He was too smart for me this time, I'll admit that—But it may be different the next time. I'll get him yet!

(*Sings:* Tune: *Toreador.*)

Hark to my warning, I will get him yet!
I'll get him yet! I'll get him yet!
I, another plan will devise. I will take his crown!
Perhaps, this very day, I'll bring him down!
AH! I will get him yet!

Anne: (*As Jane finishes last bar*) Look out, Jane, or he'll get you now. Here he comes!

(*They take their seats. The bell rings and Prof. Math calls the class to order.*)

Prof. Math: Before I go on with the advance work, I will devote a little time to answering questions on today's assignment. Are there any questions? Yes, Jane, what is your question?

Jane: It isn't about the assignment, Professor,—It's a more general question. Would you mind telling us what good this course is. What practical value it has for us?

Prof. Math: How often have I heard that question!

(*Sings:* Tune: *Yankee Doodle, with one slight change in the music.*)

'Tis absolute futility, to question my utility.
I never have to work. I never have to work.
I have no reality, in sense or sensuality,
And mundane tasks, I shirk, and mundane tasks I shirk.
The stupid questions, students ask!
So sure they're taking me to task:
What good I do? And am I true?
Such woeful ignorance!
I am pure non-SENSE—(*Interrupted by Jane who springs to her feet shrieking dramatically.*)

Jane: He's NONsense! He admits it himself! He's NONsense! I knew it all the time!

(*Chorus of excited cries NONsense! NONsense!*)

Prof. Math (*making his voice heard above the clamor*)*:* I said "Non-SENSE." (*His words make no impression on the excited students.*)

Jane (*chanting dramatically*)*:*

He's nonsense! He's nonsense! I knew it all the time!

Anne: But he handed such a line!

Ruth: And such awful condescension—
We thought our comprehension
Was in error.

Jane: He's worked us to the limit.
Every hour, every minute,
Spent in terror
That we would flunk the course.

Chorus: He's worked us to the limit
Every hour, every minute,
Spent in terror

That we would flunk the course,
That we would flunk the course.

Chorus: Tune: Farmer in the Dell.

But we won't flunk the course, No, we won't flunk the course.
HI, HO, we'll lay him low. There won't be any course!

(*The group crowds around Math, concealing him from the audience. When they spread out again, Math is seen lying motionless, on the floor.*)

There isn't any Math. There isn't any Math.
HI, HO, we've laid him low. There isn't any Math!

We've won the victory! We've won the victory!
HI, HO, we've laid him low, and set all students free!

We've set all students free, we've set all students free!
HI, HO, away we go. We've set all students free!

(*Serpentine off the stage, still singing.*)

(*Math comes to life, and rises slowly to his feet.*)

Math: Dead, am I? So they think! Poor deluded creatures,—will they never learn that I cannot die—I, who am immortal? Crushed to the earth, I rise again, as always I shall rise. (*Pauses, and walks up and down, meditating.*) So they have abandoned me.—Well, let them go. They never spoke my language! *MY* language? They can't speak the English language. That was the trouble. Yes, they've gone—but they will be back. As soon as they find that they need my help, one by one back they'll come, these bread and butter students. But not one will come seeking me for my own sake. Not one? Ah, I am wrong—I forgot, always there are EXCEPTIONS! I won't be deserted long.—But for the present, I am alone.

(*Sings: Tune: Tit Willow.*)

In this world of my own, I am never alone.
And time does not drag unduly.
My beautiful laws admit of no flaws,
And no one is ever unruly.
Though wars wage without, and the battle cries shout,
Here Order holds sway, undisputed.
No factional strife makes a discord of life,
For each has his limits computed.

(*As Math finishes his song, the students are heard approaching, cheering.*)

Math: They are coming back! I'll play dead a little longer. It would be a pity to spoil their fun.

(*The students serpentine on to the stage, shouting and cheering.*)

Chorus: (*Tune: Whistle While You Work.*)

We've done away with Math. We have swept him from our path.
We've laid him low, and now we go, to celebrate in song.
Oh, sing the whole day long! Oh, sing a joyful song!
Good news we bring. Rejoice and sing,
And let the echoes ring!
No Algebra to do, no Trig to bother you,
No Differential Calculus
To spoil the lives of all of us—

We've done away with Math. We have swept him from our path.
OH, HO, HO, HO— Away we go.
We've done away with Math.

(*As the students finish singing, an excited group rushes upon the stage. They are: The Dean, Physics, Chemistry, Business, and the Bursar.*)

The Dean: Stop this noise, immediately, and take your seats. What is this all about? It can't be true—that nonsense Helen told me—that Math is dead! Oh, (*catching sight of Math, lying on the floor*) it *is* true! Math *is* dead! What are we going to do?

Physics: I don't know what *you're* going to do. Speaking for myself, I might as well shut up shop, and go home. I can't get along without Mathematics.

Chemistry: Don't expect much work from *me*, either. I need Mathematics to help me in my line.

Business: And me—What am I going to do? I tell you—I'll go bankrupt without Math!

The Dean (*wringing her hands*)*:* Oh, what are we going to do? Jane, you wicked girl,—you are responsible for this terrible situation. You are expelled—pack up your things, and go home!

The Bursar: You might as well face facts, Dean—we'll all have to pack up and go home—I can't run the college without the help of Mathematics.

Anne: Oh, look, look! Who is this coming?

Ruth: Why it is the World—and he is lame! He's wearing crutches!

Jane: Oh, what have I done—what have I done! Why didn't someone tell me Math was so important?

(*The World reaches the center of the stage. He wears a large globe over his head, and supports himself on crutches.*)

Jane: (*Tune: Robin Adair.*)

Who has done this to thee? Poor crippled World.
Thy wings are torn from thee, poor crippled World.
Oh weep and mourn with me, because all hope is fled.
The World on crippled feet, for Math is dead!

(*Math rises to his feet. Excited cries, and cheers as the group realizes that he is alive.*)

Prof. Math: Cheer up, World, I am not dead. The report of my death was greatly exaggerated. I am very much alive, as you see. I must admit that it makes me very happy to see you all so happy about that fact.—Jane, you have done me a service. Thanks to you, Mathematics comes into his own, at last. I thank you—and by the way I'll thank you to surrender my crown.

Jane: Take your crown, King Mathematics. Truly you are a King.

Chorus: (*Tune: Melody in F.*)

King Mathematics comes into his own.
Our hearts enthrone him, Our voices own him,
Beautiful dreamer, whose dreams oft come true,
Moulding the World anew.
So great his powers, the gifts that he brings,
Because of him, the World has wings.
Marvels of Nature forever unfurled,
Truly He rules the World.

The End.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

E 349. *Proposed by R. H. Bardell, University of Wisconsin.*

Show that for all positive integral values of k,

$$\sum_{n=0}^{k}(-1)^nC_n^k/(2n+1)=\frac{2^k\cdot k!}{(2k+1)(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}.$$

E 350. *Proposed by V. Thébault, Le Mans, France.*

If from the feet of the bisectors of the interior angles of a triangle perpendiculars erected to the respective sides are concurrent, prove that the triangle is isosceles.

E 351. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Determine b as a function of a such that 4 may be the sum of the squares of the roots of the equation, $x^3+ax^2+bx=2a$, and discuss the equation, considering a as a variable parameter.

SOLUTIONS

E 312 [1938, 47]. *Proposed by D. L. MacKay, Evander Childs High School, N.Y.*

If the scalene triangle ABC has its external angle bisectors at B and C equal, show that $(s-a)/a$ is the geometric mean of $(s-b)/b$ and $(s-c)/c$.

Solution by E. P. Starke, Rutgers University.

Let the external bisector of angle C meet AB in M. Either A lies between B and M, or B lies between A and M. In the first case, angle M will equal $\frac{1}{2}\pi-(B+\frac{1}{2}C)$. In the second case, angle M equals $B+\frac{1}{2}C-\frac{1}{2}\pi$. Thus $\sin M=\pm\cos(B+\frac{1}{2}C)$. The bisector CM could not be parallel to AB, for then we would have $\frac{1}{2}C+B=\frac{1}{2}\pi$, and angles B and A would be equal, contrary to hypothesis.

In triangle BCM we have, $CM/BC=\sin B/\sin M=\pm\sin B/\cos(B+\frac{1}{2}C)$. Similarly, from triangle BCN, N being the intersection of AC with the external bisector of angle B, we have $BN/BC=\pm\sin C/\cos(C+\frac{1}{2}B)$. Since CM is given equal to BN, we must have

$$\sin C\cos(B+\tfrac{1}{2}C)\mp\sin B\cos(C+\tfrac{1}{2}B)=0.$$

When reduced to functions of half angles, this becomes

$$\text{(A)} \qquad 2[\sin \tfrac{1}{2}B \mp \sin \tfrac{1}{2}C][\cos^2 (\tfrac{1}{2}B + \tfrac{1}{2}C) \pm \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C] = 0.$$

If both bisectors fall under the same category or case, the upper signs in (A) will apply. Since the second factor of (A) is then the sum of the two positive quantities, it cannot vanish, so that $\sin \frac{1}{2}B = \sin \frac{1}{2}C$, and the triangle ABC is isosceles. But if the two bisectors fall in opposite categories or cases, the lower signs in (A) apply, the first factor of (A) cannot vanish, and we must have $\sin \frac{1}{2}B \sin \frac{1}{2}C = \cos^2 (\frac{1}{2}B + \frac{1}{2}C) = \sin^2 \frac{1}{2}A$. This is equivalent to the proposed relation, as may be shown by substituting $[(s-b)(s-c)/bc]^{1/2}$ for $\sin \frac{1}{2}A$, and analogous formulas for the other sines. Furthermore, this condition is sufficient as well as necessary.

Also solved by L. M. Kelly and the proposer.

E 313 [1938, 47]. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Find four rational numbers such that their sum, the sum of their squares, and the sum of their cubes, each equals 2.

Solution by A. G. Clark, Fort Collins, Colorado.

The symmetric functions, $\sum a$, $\sum a^2$ and $\sum a^3$ each being equal to 2 gives us also that $\sum ab = 1$, and $\sum abc = 0$. Hence all rational solutions of the problem will be given by sets of rational roots of the quartic equation,

$$x^4 - 2x^3 + x^2 = R^2,$$

where R is rational, since the left member of this equation is a perfect square. But this quartic degenerates into the two quadratics,

$$x^2 - x \pm R = 0.$$

The necessary and sufficient conditions that all roots be rational are evidently that $1+4R = p^2/r^2$ and $1-4R = q^2/r^2$, where p, q, and r are integers. If the above equations are added, we have $p^2+q^2 = 2r^2$, which is a well known equation whose solutions are all given by

$$p = k(m^2 + 2mn - n^2), \qquad q = k(m^2 - 2mn - n^2), \qquad r = k(m^2 + n^2).$$

Here m and n are relatively prime integers, and k is any integer. Substituting these values above gives us the four fractions, $(m^2+mn)/(m^2+n^2)$, $(m^2-mn)/(m^2+n^2)$, $(n^2+mn)/(m^2+n^2)$, and $(n^2-mn)/(m^2+n^2)$, as representing all rational solutions of the problem.

Also solved by F. A. Alfieri, Fred Discepoli, Daniel Finkel, A. D. Maxwell, W. C. Rufus, E. P. Starke, and Samuel Zuckerman.

E 314 [1938, 47]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Find the locus of the center of a variable sphere which cuts each of two (or three) fixed planes in a circle of constant size.

No solutions have been received for this problem.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3891. *Proposed by J. R. Musselman, Western Reserve University.*

The Apollonian circles of a triangle meet in two points, the Hessian points h_1 and h_2. Show that the two Beltrami points (inverses in the circumcircle of the Brocard points) form with either Hessian point an equilateral triangle. Hence each Beltrami point is the center of the circle passing through h_1 and h_2 and the other Beltrami point. Naturally a Brocard point lies on each circle.

3892. *Proposed by George Rutledge, Massachusetts Institute of Technology.*

Show that, for all even values of n,

$$\int_0^1 \frac{1}{u}\left\{\left(\log u + \log\frac{1+u}{1-u}\right)^n - (\log u)^n\right\}du = 0.$$

3893. *Proposed by Norman Anning, University of Michigan.*

From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. Prove that the number of essentially different possible triangles is the integer nearest to $n^2/12$.

3894. *Proposed by Walter Leighton, The Rice Institute.*

Given a polynomial equation

$$(1)\qquad f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0,$$

where the a_i are rational integers, find all roots of the form $(a \pm \sqrt{b})/c$, where a, b, c are integers, $c \neq 0$, and $\sqrt{b}$ is irrational (possibly imaginary).

3895. *Proposed by V. Thébault, Le Mans, France.*

Four spheres (ω_1), (ω_2), (ω_3), (ω_4) pass respectively through the vertices A_1, A_2, A_3, A_4 of a tetrahedron and intersect in pairs on the corresponding edges. Straight lines A_5A_1, A_5A_2, A_5A_3, A_5A_4 which join an arbitrary point A_5 to the vertices of the tetrahedron cut the respective spheres again in α_1, α_2, α_3, α_4 and these four points with A_5 lie on a sphere (ω_5). Continuing in this way, the lines $A_nA_1, A_nA_2, \cdots, A_nA_{n-1}$ which join an arbitrary point A_n to the points of a preceding set $A_1, A_2, \cdots, A_{n-1}$ cut the respective spheres (ω_1), (ω_2), $\cdots$, (ω_{n-1}) in the points $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ which with A_n lie on a sphere (ω_n). (1) Show that the spheres (ω_1), (ω_2), $\cdots$, (ω_n) meet in a point P. (2) Show that the points diametrically opposite to $A_1, A_2, \cdots, A_n$ on the corresponding spheres lie in a plane through P.

Note by the Proposer. The part (2) is an extension of a theorem by Bouvaist-Delens, *Annales de la Société Scientifique de Bruxelles*, 1937, p. 159.

SOLUTIONS

3780 [1936, 245]. *Proposed by J. M. Feld, New York, N. Y.*

In triangle $A_1A_2A_3$ the transversal A_iD_i divides A_jA_k in the ratio $A_jD_i:D_iA_k=p_i:q_i$, where ijk is a cyclic permutation of 123. The transversals A_iD_i and A_jD_j intersect in P_k. Find the value of the cross ratio

$$\frac{P_3P_2}{P_2A_1}\bigg/\frac{P_3D_1}{D_1A_1}$$

in terms of the p's and q's. Show that Ceva's theorem is a special case. (*An omission supplied.*)

I. *Solution by Maud Willey, Long Beach, Miss.*

Let A_2P_2 cut A_1A_3 in B. The theorem of Menelaus applied first to triangle $A_1A_2D_1$ and the transversal $D_3P_2A_3$, and then to $A_1D_1A_3$ and transversal A_2P_2B, gives $A_1D_3\cdot A_2A_3\cdot D_1P_2=D_3A_2\cdot D_1A_3\cdot P_2A_1$ and $A_1P_2\cdot A_2D_1\cdot A_3B=P_2D_1\cdot A_2A_3\cdot BA_1$. Hence $A_1B/BA_3=p_1p_3/q_1q_3$. The point ranges $A_1P_2P_3D_1$ and $A_1BD_2A_3$ are in perspective from A_2; and hence $A_1P_2\cdot P_3D_1/P_2D_1\cdot A_1P_3=A_1B\cdot D_2A_3/BA_3\cdot A_1D_2=p_1p_2p_3/q_1q_2q_3$. This gives for the ratio in the problem $1-(q_1q_2q_3/p_1p_2p_3)$.

If two of the P points coincide this last ratio is zero and we have Ceva's theorem.

II. *Solution by J. H. Butchart, Phillips University, Enid, Okla.*

Let the vectors of A_1, A_2, A_3 be 0, $\mathbf{a}_2$, $\mathbf{a}_3$, and set $p_i+q_i=1$. Then the vectors of D_1, P_2, P_3 are respectively $q_1\mathbf{a}_2+p_1\mathbf{a}_3=\mathbf{d}$; $p_3\mathbf{d}/(p_1p_3+q_1)$; $q_2\mathbf{d}/(q_1q_2+p_1)$. The cross ratio in the problem reduces to $1-(q_1q_2q_3/p_1p_2p_3)$ after certain simplifications. If P_2 and P_3 coincide, this ratio vanishes and we have Ceva's theorem.

Solved also by Elsie J. McFarland, J. Rosenbaum, F. Underwood, William Wernick, and the proposer.

3802 [1936, 580]. *Proposed by D. H. Lehmer, Lehigh University.*

Let $0, u_1, u_2, u_3, \cdots$ be a sequence of numbers satisfying the recurrence $u_{n+1}=au_n+bu_{n-1}$, and consider the function

$$f(x)=\sum_1^\infty u_nx^n/n!.$$

Show that $f(x)=-e^{ax}f(-x)$.

Solution by M. T. Bird, Utah State Agricultural College

The function $y=f(x)$ is the unique solution of the linear differential equation

$$y'' - ay' - by = 0$$

with the initial conditions $y(0)=0$, $y'(0)=u_1$.

Direct substitution reveals that $y=-e^{ax}f(-x)$ is likewise a solution of the indicated differential equation and initial conditions. Consequently, we must have

$$f(x) \equiv - e^{ax}f(- x).$$

The problem can also be solved by exhibiting $f(x)$ in terms of exponentials. However, this method requires a division into two cases.

Solved also by T. S. George, B. P. Hoover, F. Underwood, and W. M. Whyburn.

Editorial Note. In the direct substitution referred to above it may aid the reader to note first that $f(-x)$ satisfies the differential equation $y''+ay'-by=0$, where $f(x)$ is a solution of $y''-ay'-by=0$. The latter equation and $u_{n+2}-au_{n+1}-bu_n=0$ have the same characteristic equation $x^2-ax-b=0$; and there are two cases, one for unequal roots and the other for equal roots. Whyburn set $w'=f(x)$ with $w(0)=0$ and found that w is the unique solution of $w''-aw'-bw=u_1$, $w(0)=0$, $w'(0)=0$. After obtaining the solution of this equation by standard methods he verified the desired result. The remaining solutions were essentially as follows for the two cases:

$$u_n = u_1(\alpha^n - \beta^n)/(\alpha - \beta), \qquad u_n = u_1 n\alpha^{n-1},$$

where α and β are the roots of the characteristic equation. The verification of the desired result for the two cases is then easy. Whyburn obtained

$$f(x) = \frac{2u_1e^{ax/2}}{d} \sinh\left(\frac{dx}{2}\right), \qquad d = \alpha - \beta,$$

which covers both cases, if we take the limit for $d \to 0$ when the two roots are equal.

3803 [1936, 580]. *Proposed by Frank Ayres, Jr., Dickinson College.*

Prove that

$$\text{(a)} \qquad \sum_{s=0}^{n} (-1)^s 2^{2s}\binom{n}{s}\binom{2n-2s}{n-s} = (-1)^n\binom{2n}{n},$$

$$\text{(b)} \qquad \sum_{s=0}^{n-1} (-1)^s 2^{2s}\binom{n}{s}\binom{2n-2s}{n+1-s} = (-1)^{n-1}\binom{2n}{n+1}.$$

Solution by Emma Lehmer, Bethlehem, Pa.

Parts (a) and (b) of this problem can be thought of as special cases for $k=n$ and $k=n+1$ of the following identity.

$$\text{(1)} \qquad \sum_{s=0}^{m} (-1)^s 2^{2s}\binom{n}{s}\binom{2n-2s}{k-s} = (-1)^k\binom{2n}{k},$$

where m is the lesser of n and $2n-k$.

To establish (1) we start with the obvious algebraic identity

$$(1+x)^{2n}\left[1-\frac{4x}{(1+x)^2}\right]^n=(1-x)^{2n},$$

and first expand the binomial in square brackets by the binomial theorem. Then multiplying the expansion by $(1+x)^{2n}$, we obtain

$$\sum_{\nu=0}^{n}(-1)^{\nu}2^{2\nu}\binom{n}{\nu}(1+x)^{2n-2\nu}x^{\nu}=(1-x)^{2n}.$$

We next expand the remaining binomials and get

$$\sum_{\nu=0}^{n}(-1)^{\nu}2^{2\nu}\binom{n}{\nu}x^{\nu}\sum_{\mu=0}^{2n-2\nu}\binom{2n-2\nu}{\mu}x^{\mu}=\sum_{n=0}^{2n}(-1)^{n}\binom{2n}{m}x^{m}.$$

It now remains to equate the coefficients of x^k in the above identity. To do this let $\nu=s$, $\mu=k-s$, and $m=k$ and sum over s running from 0 to n if $k\leqq n$ and from 0 to $n-2k$ if $k>n$. This gives (1).

3804 [1936, 581]. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres with non-coplanar centers, their six spheres of similitude are grouped into three pairs so that each pair involves all four given spheres. If the spheres in two of these pairs are orthogonal, the two spheres of the third pair are also orthogonal to each other. (See, for instance, *Modern Pure Solid Geometry* by the proposer. The Macmillan Co., 1935.)

Solution by the Proposer.

The centers of the six spheres of similitude of four given spheres taken in pairs lie in a plane (P) and on the six edges of the tetrahedron $ABCD$ having for vertices the centers, A, B, C, D of the given spheres (*ibid.*, p. 204, art. 635); hence these six centers are the vertices of a complete quadrilateral (Q). The six spheres of similitude are orthogonal to sphere $(O)=ABCD$, hence the plane (P) cuts these spheres of similitude along circles orthogonal to the circle of intersection (c) of (P) with (O). Now if the two spheres of similitude having their centers on the two opposite edges AB, CD of the tetrahedron $ABCD$ are orthogonal to each other, their centers are conjugate with respect to the sphere (O) (*ibid.*, p. 181, art. 576) and therefore also with respect to the circle (c).

Thus, by hypothesis, two pairs of opposite vertices of the complete quadrilateral (Q) are conjugate with respect to (c), hence the same is true about the third pair of vertices, according to Hesse's theorem. Consequently the two spheres of similitude having these two vertices for centers are orthogonal to each other (*ibid.*, p. 181, art. 573).

Editorial Note. A solution by vectors can be given. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ be the vectors of the vertices of a tetrahedron with the origin O at the circumcenter, and let r_1, r_2, r_3, r_4 be the radii of spheres having their centers at the respective vertices. If R is the radius of the circumsphere, $R^2 = \mathbf{a}_i^2$, $i = 1, 2, 3, 4$; and then the equations of two spheres and their sphere of similitude are respectively

$$\begin{gathered} \mathbf{x}^2 + R^2 - 2\mathbf{a}_1 \cdot \mathbf{x} - r_1^2 = 0; \qquad \mathbf{x}^2 + R^2 - 2\mathbf{a}_2 \cdot \mathbf{x} - r_2^2 = 0; \\ (r_1^2 - r_2^2)(\mathbf{x}^2 + R^2) - 2(r_1^2 \mathbf{a}_2 - r_2^2 \mathbf{a}_1) \cdot \mathbf{x} = 0. \end{gathered} \tag{1}$$

The sphere of similitude for 3, 4 is then easily written. A necessary and sufficient condition that the spheres of similitude 1, 2 and 3, 4 be orthogonal is the vanishing of

$$\phi(12, 34) \equiv (r_1^2 \mathbf{a}_2 - r_2^2 \mathbf{a}_1) \cdot (r_3^2 \mathbf{a}_4 - r_4^2 \mathbf{a}_3) - (r_1^2 - r_2^2)(r_3^2 - r_4^2) R^2. \tag{2}$$

It is easily verified that

$$\phi(12, 34) + \phi(13, 42) + \phi(14, 23) \equiv 0; \tag{3}$$

hence, if the two spheres in two of these pairs are orthogonal, the two spheres of the remaining pair must also be orthogonal.

If the radii of the four spheres are given, the orthogonality requirement of the problem imposes two conditions upon the relative positions of the centers. The simplest case is that for which all the radii are equal: it then follows from (2) and (3) that $(\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{a}_k - \mathbf{a}_l) = 0$ for all sets of values of the subscripts no two of which are equal. This says that the tetrahedron of centers is orthocentric.

3799 [1936, 500]. *Proposed by E. G. Olds, Carnegie Institute of Technology.*

From an urn containing ten balls, numbered from one to ten, balls are drawn one by one and placed in a row of holes, numbered from one to ten, each ball being placed in the proper hole. What is the probability that there will not be an empty hole between two filled ones at any time of the drawing?

I. *Solution by R. F. Rinehart, Case School of Applied Science.*

There is a one to one correspondence between the orders of filling a board of n holes, without the occurrence of a vacant hole between two filled ones at any point in the proceedings, and the orders of emptying a full n hole board under the same restriction. The number of orders of so emptying an n hole board is clearly 2^{n-1}. Hence the required probability is $2^{n-1}/n!$.

A second method is as follows: The number of distinct ways of filling a board with $n+1$ holes is twice the number of ways for an n hole board, since there are precisely two distinct ways of having a last hole to be filled under the required condition. For a one hole board the number of ways is 2^0. Hence the number of ways for a n hole board is 2^{n-1}.

II. *Solution by Michael Goldberg, Washington, D. C.*

Let there be n balls and n holes. Suppose the first ball is placed in the kth hole. Then if no gap is to be formed in the array, each successive ball must be placed directly adjacent, on one side or the other, of the unbroken sequence of balls in their holes. This is the same as the number of permutations of $n-1$

objects of which $k-1$ are of one type and the remaining $n-k$ are of another type. This number is $(n-1)!/(k-1)!(n-k)!$. Hence the total number of successful drawings is

$$\sum_{k=1}^{n} (n-1)!/(k-1)!(n-k)! = 2^{n-1}.$$

The probability of making a complete drawing without gaps is $2^{n-1}/n!$. For $n=10$ it is $2^9/10!$ or $2/14175$.

Solved also by J. P. Ballantine, W. B. Campbell, C. Eisenhart, H. M. Gehman, H. O. Hanson, G. B. Lang, Philip McCarthy, Joseph Noback, Ellis R. Ott, and the proposer.

3801 [1936, 580]. *Proposed by D. H. Lehmer, Lehigh University.*

Show that

$$\operatorname{arccot} 1 = \operatorname{arccot} 2 + \operatorname{arccot} 5 + \operatorname{arccot} 13 + \operatorname{arccot} 34 + \cdots,$$

where these integers constitute every other term of the Fibonacci series and satisfy the recurrence $u_{n+1}=3u_n-u_{n-1}$.

Solution by M. A. Heaslet, San Jose State College, San Jose, Calif.

The Fibonacci sequence is determined by $v_{n+2}=v_{n+1}+v_n$ with $v_1=v_2=1$. We shall show that

$$\text{(1)}\qquad \operatorname{arccot} v_2 - \operatorname{arccot} v_3 - \operatorname{arccot} v_4 - \cdots - \operatorname{arccot} v_{2n+1} = \operatorname{arccot} v_{2n+2},$$

where the principal value of arccot is meant. We have

$$\begin{aligned}\operatorname{arccot} v_{2n} - \operatorname{arccot} v_{2n+1} &= \operatorname{arccot}\left[\frac{v_{2n}v_{2n+1}+1}{v_{2n+1}-v_{2n}}\right]\\ &= \operatorname{arccot}\left[\frac{v_{2n}v_{2n+1}+1}{v_{2n-1}}\right].\end{aligned}$$

From the easily deduced relation $v_{m+1}v_{m+2}-v_mv_{m+3}=(-1)^m$ (see *Théorie des Nombres*, N. Kraitchik, page 3) the above becomes on setting $m=2n-1$,

$$\text{(2)}\qquad \operatorname{arccot} v_{2n} - \operatorname{arccot} v_{2n+1} = \operatorname{arccot} v_{2n+2}.$$

Summing the members of (2) for $n=1, 2, \cdots, n$, we obtain (1). Since the right member of (1) approaches zero as n becomes infinite, we have finally

$$\text{(3)}\qquad \operatorname{arccot} v_2 = \sum_{n=1}^{\infty} \operatorname{arccot} v_{2n+1}.$$

If we set $v_{2n+1}=u_{n+1}$, $v_2=v_1=u_1=1$, we may write this result as

$$\text{(4)}\qquad \operatorname{arccot} u_1 = \sum_{n=1}^{\infty} \operatorname{arccot} u_{n+1},$$

where $u_1=1$, $u_2=2$, $u_{n+2}=3u_{n+1}-u_n$, as is easily verified.

Solved also by L. S. Johnston, B. P. Hoover, and E. P. Starke.

Editorial Note. In Johnston's solution arccot x was denoted by $C(x)$ and the formula

$$C(v_{2m}) = C(v_{2m+1}) + C(v_{2m+3}) + \cdots + C(v_{2m+2k-1}) + C(v_{2m+2k})$$

was deduced. This can be written

$$C(v_{2m}) = \sum_{k=1}^{\infty} C(v_{2m+2k-1}).$$

Another result was obtained by denoting the inverse of coth x by $H(x)$ and showing that

$$H(v_{2n+1}) - H(v_{2n+2}) = H(v_{2n+3}), \qquad n \geqq 1,$$

and then that

$$H(v_{2m+1}) - H(v_{2m+2}) - H(v_{2m+4}) - \cdots - H(v_{2m+2k}) = H(v_{2m+2k+1}), \quad m \geqq 1.$$

This last result is proved by using the functional relation

$$H(v_{2n+1}) - H(v_{2n+2}) = H\left[\frac{v_{2n+1}v_{2n+2} - 1}{v_{2n+2} - v_{2n+1}}\right],$$

and $v_{2n+1}v_{2n+2}-v_{2n}v_{2n+3}=1$.

A general theorem was proved for which the above are special cases. Let the sequence $t_1, t_2, \cdots$ satisfy $t_{n+2}=t_{n+1}+t_n$, $t_3t_1-t_2=a^2$, where a^2 is not necessarily positive. Then any function $F(t)$ which satisfies

$$F(t_n) - F(t_{n+2}) = F\left[\frac{t_{n+2}t_n + (-1)^n a^2}{t_{n+2} - t_n}\right]$$

also satisfies $F(t_n)=F(t_{n+1})+F(t_{n+2})$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The Mathematical Association has for sale at 50 cents a copy a few copies of the following numbers of *Runkle's Mathematical Monthly*: Volume II: March, April, June, July, and September 1860 (with index), mostly lacking covers. Also one copy each of Volume III, October 1860, and March 1861, for 60 cents each. These may be had by sending to Secretary W. D. Cairns.

Several Benjamin Peirce Instructorships at Harvard University are open for the academic year 1939–40. These instructorships are ordinarily awarded to men who have recently received the Ph.D. degree or have had equivalent training.

Those interested in applying should write to the Chairman of the Division of Mathematics.

There will be a meeting of the Chapters of Pi Mu Epsilon at Richmond Virginia on Wednesday December 28 at 8:00 A.M. in the Hotel William Byrd.

Professor E. T. Bell of the California Institute of Technology has been awarded the gold medal of the Commonwealth Club of California for his book entitled *Men of Mathematics*. The medal is awarded for "the best scholarship and research work published during 1937."

The University of St. Andrews conferred an honorary degree upon Dean G. D. Birkhoff of Harvard University on July 5, at the tercentenary celebration of the birth of James Gregory.

A dinner was given in honor of Professor Emeritus Virgil Snyder and Mrs. Snyder on Friday evening, September 9, at the Hotel Woodstock, New York. The number attending was thirty-six. A silver bowl was presented on which is engraved: "Presented to Professor Virgil Snyder, Cornell University, by his Ph.D.'s as a token of appreciation," and also the names of the thirty-nine mathematicians who have written doctoral theses with Professor Snyder.

Assistant Professor Reinhold Baer of the University of North Carolina has been appointed an associate professor at the University of Illinois.

D. C. Baillie has been appointed a lecturer at the University of Toronto.

Dr. R. H. Bardell of the University of Wisconsin Extension Division at Milwaukee has been promoted to an assistant professorship.

Dr. R. P. Boas, Jr., National Research Fellow in mathematics, will spend the year 1938–39 at the University of Cambridge.

On an exchange arrangement, Dr. A. W. Boldyreff has been appointed an associate professor at the University of Illinois for the year 1938–39, while Dr. O. K. Bower is an assistant professor at the University of Arizona.

Assistant Professor H. E. Bray of Rice Institute has been promoted to an associate professorship.

Dr. Randolph Church of the United States Naval Academy has been promoted to an assistant professorship.

W. G. Cochran of Rothamsted Experimental Station in Harpenden, England, has been appointed visiting lecturer in mathematics and statistics at Iowa State College for the fall quarter of 1938.

Associate Professor L. W. Cohen of the University of Kentucky is on leave of absence for 1938–39 to study at Brown University.

Associate Professor C. H. Currier of Brown University has retired with the title emeritus.

Professor Arnaud Denjoy of the University of Paris is in residence as an exchange professor at Harvard University for the first half of 1938–39.

Dr. A. H. Diamond of the University of California at Los Angeles has been appointed professor and head of the department of mathematics at Oklahoma Agricultural and Mechanical College.

Dr. C. H. Dowker has been appointed lecturer at the University of Western Ontario.

Dr. Nelson Dunford of Yale University has been promoted to the rank of assistant professor.

Dr. W. H. Erskine of Bethany College, West Virginia, has been promoted to an assistant professorship.

Assistant Professor H. P. Evans of the University of Wisconsin has been promoted to an associate professorship.

Associate Professor D. A. Flanders of New York University has returned after a year's study at the University of Copenhagen.

Dr. J. S. Frame of Brown University has been promoted to an assistant professorship.

Assistant Professor H. J. Gay of Worcester Polytechnic Institute has been promoted to a professorship.

At the University of Florida, Assistant Professors H. H. Germond and Z. M. Pirenian have been promoted to associate professorships.

Associate Professor B. P. Gill of the College of the City of New York has been promoted to a professorship.

W. L. Graves of the University of Pennsylvania has been appointed an assistant professor at Drury College.

Dr. J. A. Greenwood of Duke University has been promoted to an assistant professorship.

Dr. D. C. Harkin of Alabama Polytechnic Institute has been appointed tutor at Brooklyn College for 1938–39.

Dr. M. L. Hartung of Ohio State University has been appointed an assistant professor in education at the University of Chicago.

Associate Professor T. W. Hatcher of Virginia Polytechnic Institute has been promoted to a professorship.

Dr. E. A. Hedburg of the University of Missouri has been appointed to a professorship at Baylor University.

Associate Professor G. A. Hedlund of Bryn Mawr College has a year's leave of absence to study at the Institute of Advanced Study.

Dr. W. L. Hutchings of Rollins College has been promoted to an assistant professorship.

Incident upon the transfer of the School of Engineering from Chapel Hill to Raleigh, Professor T. F. Hickerson and Associate Professor R. M. Trimble of the department of civil engineering have been added to the mathematics staff at the University of North Carolina.

Dr. Nathan Jacobson of the University of North Carolina has been promoted to an assistant professorship.

Professor C. G. Jaeger of Pomona College is on leave of absence for the first semester of 1938–39 for study at Brown University.

Associate Professor F. E. Johnston of George Washington University has been promoted to a professorship.

Assistant Professor B. W. Jones of Cornell University is on leave of absence and is studying at the Institute for Advanced Study.

Assistant Professor G. M. Jones of Howard University has been promoted to an associate professorship.

Dr. S. H. Kimball of the University of Maine has been promoted to an assistant professorship.

Dr. Morris Kline has returned to New York University after spending a two years' leave of absence at the Institute for Advanced Study.

Assistant Professor M. S. Knebelman of Princeton University has been appointed professor and chairman of the department of mathematics at the State College of Washington.

Assistant Professor A. H. Larsen of Illinois State Normal University is on leave of absence for study at the University of Wisconsin.

Dr. J. K. L. MacDonald has been appointed to an assistant professorship at Cooper Union.

C. E. Marshall of the University of Illinois has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

Assistant Professor Deane Montgomery of Smith College has been promoted to an associate professorship.

Assistant Professor H. S. Pollard of Miami University has been promoted to an associate professorship.

Dr. Walter Prenowitz of Brooklyn College has been promoted to an assistant professorship.

After two years' study at the Institute for Advanced Study, Dr. J. F. Randolph has returned to Cornell University with the rank of assistant professor.

Professor E. L. Rees, who has been associated with the University of Kentucky since 1907, has resigned to look after business interests and to travel.

E. K. Ritter of the University of Richmond has been promoted to an assistant professorship.

Associate Professor H. P. Robertson of Princeton University has been promoted to a professorship.

Dr. E. H. Rothe of William Penn College has been promoted to a professorship.

N. N. Royall, Jr., has been appointed to an assistant professorship at the Citadel.

Dr. O. F. G. Schilling has been re-appointed Johnston Scholar for the current year at Johns Hopkins University.

T. C. Sermon of the Michigan College of Mining and Technology has been promoted to an assistant professorship.

Assistant Professor J. H. Service of the Michigan College of Mining and Technology has been promoted to an associate professorship.

Dr. D. T. Sigley of the University of Kansas City has been appointed an assistant professor at Kansas State College.

Professor T. M. Simpson has been appointed acting dean of the Graduate School at the University of Florida.

Assistant Professor F. C. Smith of the College of St. Francis has been promoted to a professorship.

Assistant Professor C. E. Springer of the University of Oklahoma has a year's leave of absence for study at the University of Chicago.

Assistant Professor C. G. Stipe of the Michigan College of Mining and Technology has been promoted to an associate professorship.

Dr. Alvin Sugar of the University of California has been appointed head of the department of mathematics and physics at St. Francis College, Loretto, Pa.

Dr. C. C. Torrance of Case School of Applied Science will be a member of the Institute for Advanced Study during 1938–39.

Professor M. O. Tripp of Wittenberg College has retired with the title of Professor Emeritus.

Professor H. S. Vandiver of the University of Texas is on leave of absence for the first semester of 1938–39.

Assistant Professor W. G. Warnock of Fort Hays Kansas State College has been appointed to an assistant professorship at the University of Alabama.

Dr. K. W. Wegner of the University of Minnesota has been appointed to a professorship at the College of St. Catherine.

Assistant Professor S. S. Wilks of Princeton University has been promoted to an associate professorship.

Dr. C. P. Winsor has been appointed an assistant professor of mathematics and statistics at Iowa State College.

The following appointments to instructorships have been announced:

University of Alabama: Dr. C. R. Cassity
Agricultural and Mechanical Junior College, Arkansas: C. V. Wetzig
Brooklyn College: Dr. J. J. DeCicco
Brown University: Dr. S. E. Warschawski
University of Cincinnati: Dr. W. C. Taylor
College of the City of New York: Dr. Max Shiffman
Columbia University: Dr. J. V. Wehausen
Cornell University: Dr. Fritz Herzog, Dr. S. Saslaw, Edward Stephany
Crosby-Ironton Junior College, Minnesota: J. D. Novak
University of Florida: T. C. George, E. C. Phillips
Hunter College: Dr. Madeline Levin, Dr. Annita Tuller
University of Illinois: Dr. P. R. Halmos, Dr. Olaf Helmer
Kansas State College: Robert Rawhouser
Keuka College: H. F. Archibald
University of Kansas: Edison Greer
University of Kentucky: J. M. Boswell, V. W. Pfeiffer
Michigan State College: Dr. G. B. Van Schaak
University of Michigan: Edith R. Schneckenburger
University of Minnesota: Dr. H. H. Campaigne
University of New Mexico: Dr. Douglas Derry
University of North Carolina: Whitfield Cobb, R. E. Smith
Ohio State University: Dr. G. E. Albert
University of Oklahoma: J. C. Van de Carr
Pennsylvania State College, Dubois Center: Dr. M. E. Shanks
University of Pittsburgh, Johnstown Center: John Breiland
Princeton University: Dr. C. B. Tompkins II
Queens College: Dr. N. A. Hall, Dr. Philip Hartman, Dr. Leo Zippin

University of Rochester: Dr. C. E. Clark
Rutgers University: R. E. Traber
San Antonio Junior College: L. H. Tulloch
Springfield Junior College, Illinois: N. W. Wells
Syracuse University: Dr. A. P. Cowgill
Texas Agricultural and Mechanical College: Dr. B. W. Brewer, Walter Jennings
Texas Technological College: Dr. C. P. Brady, Lida B. May
University of Texas: W. B. Coleman, R. E. Greenwood, Charles Moran
Vassar College: Dr. Audrey Wishard
West Virginia University: M. L. Vest
Wright Junior College, Chicago: Dr. Ruth G. Mason
Yale University: T. J. Benac, E. S. Grable.

Dr. T. W. Edmondson, professor emeritus of mathematics and former acting dean of the Graduate School of New York University, died November 4, 1938, at the age of sixty-nine. He retired from teaching in 1934, after thirty-eight years on the faculty at New York University. He was a charter member of the Mathematical Association.

THE SEMICENTENNIAL CELEBRATION OF THE AMERICAN MATHEMATICAL SOCIETY

Founded at Columbia University in 1888 as the New York Mathematical Society, the American Mathematical Society very appropriately celebrated its semicentennial anniversary by unusually interesting meetings at Columbia on September 6–9. The Society was very fortunate in having present its founder, Professor Thomas Scott Fiske, whom it honored suitably at a "Birthday Dinner." It also took this occasion to express its deep appreciation of the hospitality accorded it by Columbia University, in whose halls over half of its regular meetings have been held, and in whose buildings its central office and library have been housed for half a century.

Letters of felicitation were received from a large number of learned societies and from President Franklin D. Roosevelt and from Prime Minister W. L. Mackenzie King of the Dominion of Canada. Addressed to Dean R. G. D. Richardson, secretary of the Society, the letter from President Roosevelt reads as follows:

> Please extend my greetings to the American Mathematical Society on the occasion of the celebration of its fiftieth anniversary. I trust that genuine satisfaction will come to its members as they contemplate the contribution which the Society has made during the half century of its existence.
>
> It is sometimes difficult to comprehend the values accruing to society from mathematics. It is deeply rooted in social progress as a large part of the technological advance made in recent centuries would have been impossible had it not been for the constant refinement of the essential tool of technology—mathematics. Social sciences also owe a large debt to mathematics. Future advances in the social sciences will be largely dependent upon mathematical treatment of their data.

> I congratulate the members of the American Mathematical Society on the important contributions which mathematics has already made. I trust that the years ahead will find mathematicians making significant contributions, as in the past.

Prime Minister King's letter reads:

> I shall be pleased if you will extend to the American Mathematical Society, on the celebration of the fiftieth anniversary of its inception, my greetings and warm congratulations upon attaining this significant anniversary.
>
> It is well that we should reflect, on occasions such as this, on the extent to which the learned societies of this continent have contributed to the strength and substance of our national and international life. The work of the American Mathematical Society fills a distinguished place among those academic fellowships which have done so much to keep alive, in our institutions of learning, that integrity of thought which is one of the proudest of our common traditions.
>
> It is my hope that the deliberations of the Society's Jubilee Meeting will be attended by much good fellowship and a lively appreciation of the binding character of the academic fraternity which has so long and so happily prevailed between our two countries.

The first day of the celebration was devoted to seven sectional meetings at which over one hundred research papers were communicated. The scientific part of the remainder of the celebration consisted of ten addresses by invitation, which reviewed aspects of the development of mathematics in America during the past fifty years and indicated lines of progress for the future. The speakers and titles of their addresses were as follows:

R. C. Archibald, Brown University: History of the American Mathematical Society, 1888–1938.

G. D. Birkhoff, Harvard University: Fifty years of American mathematics.

E. T. Bell, California Institute of Technology: Fifty years of algebra in America, 1888–1938.

G. C. Evans, University of California: Dirichlet problems.

E. J. McShane, University of Virginia: Recent developments in the calculus of variations.

J. F. Ritt, Columbia University: Algebraic aspects of the theory of differential equations.

J. L. Synge, University of Toronto: Hydrodynamical stability.

T. Y. Thomas, University of California at Los Angeles: Recent trends in geometry.

Norbert Wiener, Massachusetts Institute of Technology: The historical background of harmonic analysis.

R. L. Wilder, University of Michigan: The sphere in topology.

The first of these addresses, together with a full account of the meetings, will appear in full in the January 1939 number of the *Bulletin* of the Society. In greatly amplified form Professor Archibald's paper appears as Volume I of the Society's *Semicentennial Publications*. The other nine addresses are given under the title of "Semicentennial Addresses" in Volume II. These two volumes were ready for distribution at the meetings.

The Association, without holding meetings of its own, joined in the celebration by the Society, and the following letter was formally presented:

> The Mathematical Association of America, through its officers and trustees, felicitates the American Mathematical Society on fifty years of successful activity and guidance in the promotion of mathematical research in America. It has developed as a nucleus of an earnest and increasingly able group, respected and highly esteemed at home and abroad. It has exercised a strong influence not merely in pure mathematics but in the fields of secondary and collegiate mathematics.
>
> The Mathematical Association of America is proud of the parent society and is grateful that the Society has uniformly given it a fine support and cooperation. May the next fifty years bring an even more notable advance in high and worthy scholarship.

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